Matchings between Point processes

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by

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Declaration

I hereby declare that the work of this Thesis is carried out by me in Integrated Ph.D. program under the supervision of Professor Manjunath Krishnapur, and in the partial fulfillment of the requirements of the Master of Science Degree of the Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma or any other title elsewhere.

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Abstract

In this thesis we study several kinds of matching problems between two point processes. First we consider the set of integers $\mathbb{Z}$. We assign a color red or blue with probability $\frac{1}{2}$ to each integer. We match each red integer to a blue integer using some algorithm and analyze the matched edge length of the integer zero. Next we go to $\mathbb{R}^d$. We consider matching between two different point processes and analyze a typical matched edge length. There we see that the results vary significantly in different dimensions. If X is a typical matched edge length, then in dimensions one and two ($d = 1, 2$), even $\mathbb{E}\left[X^{\frac{d}{2}}\right]$ does not exist. On the other hand in dimensions more than two ($d \geq 3$), $\mathbb{E}\left[\exp(cX^d)\right]$ exist, where c is a constant that depends on d only.

In the last three chapters we discuss about matching problems in a finite setting, namely in $[0,1]^2$. Take n red points and n blue points chosen independently and uniformly in $[0,1]^2$. There are $n!$ many possible matchings between these red and blue points. We investigate the optimal average matched edge length and the minimum of the maximum matched edge lengths, where the minimum is over all possible $n!$ matchings. There we see that the optimal average matched edge length is like $\sqrt{\frac{\log n}{n}}$ and the min-max matched edge length is like $\frac{(\log n)^{\frac{3}{4}}}{\sqrt{n}}$. In the proof of the min-max matching problem we use a technique called Generic chaining, introduced by Talagrand.

Instead of looking at $[0,1]^2$, if we look at $[0,1]^d$ for $d \geq 3$, then we can get rid of the extra $\log n$ factor in the average matched edge length and it will be just $n^{-\frac{1}{d}}$. We will not discuss this in the Thesis.

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Chapter 0

Introduction

Matching problems occur in several places. As an example consider n red points $\{X_i\}_{i\leq n}$ and n blue points $\{Y_i\}_{i\leq n}$ that are chosen according to the distributions F and G in a unit square in the plane. We would like to check whether F and G are same or not. This is a hypothesis testing problem. For that reason we try to match them up in an optimal way. In Chapter 4 we will see that if F and G are uniform then the optimal average matched edge length is $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$. One can intuitively think that if F and G are different then the optimal average matched length is likely to be higher.

In our discussion we will consider only two color matching problems in different circumstances. In $\mathbb{Z}$ we assign a color *blue* or *red* with probability $\frac{1}{2}$ to each point. We want to match red points to the blue points and analyze a typical matched edge length. In $\mathbb{R}^d$ we have two infinite sets of randomly and independently chosen points and we match them up using some algorithm that doesn't depend on the coordinates of the points, and we want to see the behavior of a typical matched length. The results vary with the dimension d.

Before moving forward, we need to introduce a mathematical formulation of these problems.

0.1 Basic definitions

Definition 0.1. (Point measure): A point measure μ_p on a locally compact separable metric space X is a locally finite, regular measure with respect to the Borel σ-algebra $\mathcal{B}(X)$, such that $\mu_p(A) \in \mathbb{N} \cup \{0, \infty\}$ for every $A \in \mathcal{B}(X)$.

We say that the support of μ_p is

$$[\mu_p] = \{x \in X : \mu_p(x) \neq 0\}.$$

Now define $M_p :=$ set of all point measures on X. We want to give a topological structure on M_p. For that purpose we define the following notion of convergence in M_p.

Definition 0.2. (Vague convergence): Let M be a set of Radon measures on a locally compact complete separable metric space X and $\{\mu_n\}_n \in M$ be a sequence of measures. We say μ_n converges to $\mu \in M$ vaguely if $\int f \, d\mu_n \to \int f \, d\mu$ for all $f \in C_c(X)$, where $C_c(X)$ is the set of all compactly supported continuous real valued functions on X.

Clearly given a locally compact separable metric space X, the vague convergence induces a topology on the set of all Radon measures on X i.e. on M. Call it *vague topology* on M.

Let $\mathcal{M}_p$ be the Borel σ-algebra on M_p, where the topology on M_p is the vague topology. Define the following

Definition 0.3. (Point process): Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A point process P on a locally compact complete separable metric space X is a measurable function $P : (\Omega, \mathcal{F}) \to (M_p, \mathcal{M}_p)$.

For example in $\mathbb{R}^2$ informally a point process can be thought of as a locally finite collection of random particles from $\mathbb{R}^2$.

Definition 0.4. (Poisson point process): A Poisson point process with intensity λ on $\mathbb{R}^d$ is a point process PP_λ on $\mathbb{R}^d$ satisfying the following two properties,

i. For two disjoint subsets $B_1, B_2 \in \mathfrak{B}(\mathbb{R}^d)$, $PP_\lambda(B_1)$ and $PP_\lambda(B_2)$ are independent.

ii. For any subset $B \in \mathfrak{B}(\mathbb{R}^d)$, $PP_\lambda(B)$ has the Poisson distribution with parameter $\lambda|B|$ i.e.

$$\mathbf{P}\left(PP_\lambda(B) = k\right) = e^{-\lambda|B|} \frac{\lambda^k |B|^k}{k!} \quad \forall\, k = 0, 1, 2, \ldots,$$

where $|B|$ is the lebesgue measure of the set B.

Now to go to the matching problems we need to know mathematically what exactly does a matching mean. Let $\mathcal{R}$ and $\mathcal{B}$ be two point processes on X. For example taking $X = \mathbb{R}^2$, we can think $\mathcal{R}$ and $\mathcal{B}$ as randomly chosen red and blue points on $\mathbb{R}^2$. The problem is to match them and analyze various properties of it. Here is the formal mathematical definition of matching.

Definition 0.5. (Matching): Given two locally finite subsets U and V of X such that card(U)=card(V), a matching M between U and V is a bipartite graph with vertex sets U and V such that each vertex has degree one. An edge in M is denoted by the ordered pair (x, y), where $x \in U$ and $y \in V$. We denote $G_{U,V}$ as the set of all matchings between U and V.

In the above definition if we remove the constraint that each vertex has degree one and allow vertices to have degree zero, then it will be different kind of matching. One may call that as *partial matching*, and if each vertex has degree one then it can be called as *perfect matching*. In our case mostly we will consider perfect matchings, and call it just matching.

Now consider $G = \cup_{U,V} G_{U,V}$, where the union is taken over all $U, V \subset X$ such that U, V are locally finite and card(U)=card(V). We want to give a σ-algebra structure on G. Take $A_1, A_2 \in \mathcal{B}(X)$, and $M \in G$. Define $M(A_1, A_2) = \{(x, y) \in M : x \in A_1, y \in A_2\}$, and the function $\Lambda_{A_1,A_2} : G \to \mathbb{R}$ as

$$\Lambda_{A_1,A_2}(M) = \text{card}(M(A_1, A_2)).$$

Define the σ-algebra $\mathcal{G}$ on G in the following way,

$$\mathcal{G} = \sigma\{\Lambda_{A_1,A_2} : A_1, A_2 \in \mathcal{B}(X)\}.$$

It means that $\mathcal{G}$ is the smallest σ-algebra on G such that the functions like Λ_{A_1,A_2} are measurable.

Definition 0.6. (Matching scheme): A two-color matching scheme $\mathcal{M}$ is a measurable map $\mathcal{M} : M_p \times M_p \to G$ such that $\mathcal{M}(U, V) \in G_{[U],[V]}$, where $[U]$, $[V]$ are supports of U and V respectively.

When the underlying space is $\mathbb{R}^d$, we define a special class of matching scheme that respect the action of translation on $\mathbb{R}^d$.

Definition 0.7. (Translation-invariant matching scheme): Take a point measure $U \in M_p$. Let U_w be the translated version of the point measure U translated by $w \in \mathbb{R}^d$. A translation-invariant matching scheme $\mathcal{M}$ is a matching scheme such that for any $U, V \in M_p$ and $w \in \mathbb{R}^d$

$$(x + w, y + w) \in \mathcal{M}(U_w, V_w) \iff (x, y) \in \mathcal{M}(U, V).$$

In other words a translation-invariant matching depends only on the relative position of the points from the point processes.

In the subsequent chapters we will consider various type of matching problems. In the next three chapters we are in infinite settings and investigate; and will analyze the typical matched edge length. After that we will be in finite setting, namely in the unit square $[0, 1]^2$ and analyze a kind of "ℓ^p norm" of matched edge sequence.

Chapter 1

Two color matchings on $\mathbb{Z}$

In this chapter we consider a matching problem in an infinite discrete setup. To each point in $\mathbb{Z}$ we assign a red or blue color with probability $\frac{1}{2}$. In other words, flip a fair coin independently at each point on $\mathbb{Z}$, if the outcome is head put red color there otherwise blue. The objective

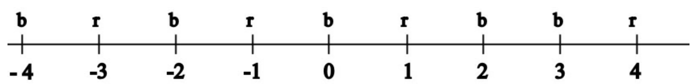


Figure 1.1: *Coin flip on $\mathbb{Z}$.*

is to match blue integers to the red integers. There are innumerable number of matching schemes but we will study only the translation-invariant matching schemes.

Example (*Meshalkin's matching algorithm*): The algorithm says that if there is a red point on the right hand side of a blue one, match blue point to that red point. Remove the matched pairs from consideration and repeat this process indefinitely for the remaining points. Clearly this is a translation-invariant matching scheme. We shall study the behavior of matched pair length for the point at origin. Let X be the random variable denoting the matched pair length for the point at origin. The following result regarding lower bound of X is proved by Holroyd and Peres [3].

Theorem 1.1. *In the above setting for any translation-invariant matching rule*

$$\mathbb{E}\left[X^{\frac{1}{2}}\right] = \infty.$$

Upper bound of X is given by the following theorem.

Theorem 1.2. *There exists a translation-invariant matching scheme such that for any $r > 0$*

$$\mathbb{P}(X > r) \leq Cr^{-\frac{1}{2}},$$

where C is a universal constant.

From Theorem 1.2 we can say that $\mathbb{E}\left[X^{\frac{1}{2}-\varepsilon}\right]$ exists for small $\varepsilon > 0$ and from Theorem 1.1 we see that $\mathbb{E}\left[X^{\frac{1}{2}}\right]$ doesn't exist. Therefore in a way these bounds are sharp.

1.1 Proof of Theorem 1.1

We will give a different proof here. The technique we will use is due to A.E. Holroyd et al. [2]. Let $\mathcal{R}[a,b]$ and $\mathcal{B}[a,b]$ respectively denote the number of red points and blue points inside the interval $[a,b]$, and $\mathcal{M}$ be a translation-invariant matching scheme. Then for $t > 0$ we have

$$\begin{aligned}
\mathbb{E}\left[\#\{x \in \mathbb{Z} \cap [0,2t] : \mathcal{M}(x) \notin [0,2t]\}\right] &\leq \mathbb{E}\left[\#\{x \in \mathbb{Z} \cap [0,2t] : |x - \mathcal{M}(x)| > x \wedge (2t - x)\}\right] \\
&= \sum_{x \in \mathbb{Z} \cap [0,2t]} \mathbb{P}[X > x \wedge (2t - x)] \\
&\leq 2 \sum_{x \in \mathbb{Z} \cap [0,t]} \mathbb{P}[X > x]
\end{aligned}$$

(since $\mathcal{M}$ is a translation-invariant matching scheme, therefore $|x - \mathcal{M}(x)| \stackrel{d}{=} |0 - \mathcal{M}(0)| = X$). Let us define discrepancy of the interval $[a,b]$ as $\mathcal{D}[a,b] := \mathcal{R}[a,b] - \mathcal{B}[a,b]$. The central limit theorem gives us

$$\frac{\mathcal{D}[0,2t]}{\sqrt{2t}} \xrightarrow{d} N(0,1).$$

Consider the function $g_K(x) = x\mathbf{1}_{\{|x| \leq K\}} + K\mathbf{1}_{\{|x| > K\}}$. This is a bounded continuous function for a fixed $K > 0$. Hence

$$\frac{1}{\sqrt{2t}}\mathbb{E}\left[(\mathcal{D}[0,2t])^+ \mathbf{1}_{\left\{(\mathcal{D}[0,2t])^+ \leq K\sqrt{2t}\right\}}\right] \to \mathbb{E}[(N(0,1))^+ \mathbf{1}_{\{(N(0,1))^+ \leq K\}}] + K\mathbb{P}\left(N(0,1)^+ > K\right),$$

as $t \to \infty$. On the other hand using Hoeffding's inequality, we see that for any $K > 0$

$$\frac{1}{\sqrt{2t}}\mathbb{E}\left[(\mathcal{D}[0,2t])^+ \mathbf{1}_{\left\{(\mathcal{D}[0,2t])^+ > K\sqrt{2t}\right\}}\right] \leq 4e^{-\frac{K^2}{2}}.$$

Also we know that $K\mathbb{P}\left(N(0,1)^+ > K\right) \to 0$ as $K \to \infty$. Combining the above estimates we get

$$\mathbb{E}\left[\left(\frac{\mathcal{D}[0,2t]}{\sqrt{2t}}\right)^+\right] \to \mathbb{E}\left[(N(0,1))^+\right] := C,$$

as $t \to \infty$. Thus we get

$$\mathbb{E}\left[\#\{x \in \mathbb{Z} \cap [0,2t] : \mathcal{M}(x) \notin [0,2t]\}\right] \geq \mathbb{E}\left[(\mathcal{R}[0,2t] - \mathcal{B}[0,2t])^+\right] \sim Ct^{\frac{1}{2}}$$

as $t \to \infty$. Therefore

$$t^{-\frac{1}{2}} \sum_{x \in \mathbb{Z} \cap [0,t]} \mathbb{P}[X > x] \geq \frac{C}{4} \text{ as } t \to \infty. \tag{1.1}$$

Now consider $f_t(x) = [t^{-\frac{1}{2}}\mathbb{P}(X > x)]\mathbf{1}_{x\leq t}$ for $x > 0$. Clearly $f_t(x) \leq f(x) := x^{-\frac{1}{2}}\mathbb{P}(X > x)$, and $f_t(x) \to 0$ as $t \to \infty$. Also we see that $\int_1^\infty f(x)\, dx \leq 2\mathbb{E}[X^{\frac{1}{2}}]$. Therefore if $\mathbb{E}[X^{\frac{1}{2}}] < \infty$ then the dominated convergence theorem gives

$$\begin{aligned} t^{-\frac{1}{2}} \sum_{x\in\mathbb{Z}\cap[0,t]} \mathbb{P}(X > x) &= t^{-\frac{1}{2}}\mathbb{P}(X > 0) + t^{-\frac{1}{2}} \sum_{x\in\mathbb{Z}\cap[1,t]} \mathbb{P}(X > x) \\ &\leq t^{-\frac{1}{2}} + \sum_{x\in\mathbb{Z}\cap[1,t]} t^{-\frac{1}{2}}\mathbb{P}(X > x) \\ &= t^{-\frac{1}{2}} + \int_1^t f_t(x)\, dx \to 0 \text{ as } t \to \infty, \end{aligned}$$

which contradicts (1.1). Hence the proof.

1.2 Proof of Theorem 1.2

This theorem can be deduced from the Meshalkin's matching scheme. The outline of the proof is given by T. Soo [5]. The coin-flips on $\mathbb{Z}$ can be thought of as a simple symmetric random walk on $\mathbb{Z}$, i.e. at each step if there is blue integer we go one step up and if there is a red integer we go one step down. Let S_m be the position of the random walk at mth step. Then

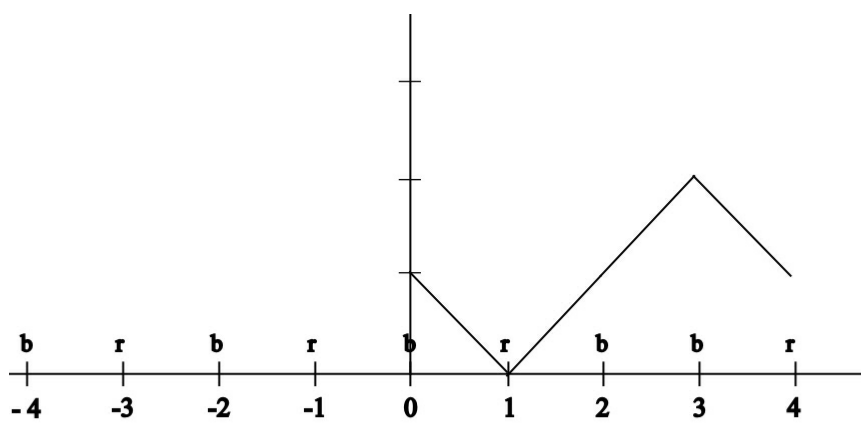


Figure 1.2: *Simple random walk on $\mathbb{Z}$.*

X i.e. the matched pair length for the point at origin is exactly the number of steps to hit zero for the first time starting from ± 1. Now let $Y_i \overset{i.i.d.}{\sim} \text{Bernoulli}(\frac{1}{2})$ (here the Bernoulli random variables are taking values ± 1) then $S_m = \sum_{i=1}^m Y_i$. Let T_0 be the time taken to hit zero for the first time, starting from ± 1. Then

$$\mathbb{P}(X = 2k + 1) = \mathbb{P}(T_0 = 2k + 1).$$

Without loss of generality let us assume that the random walk starts at 1. Then we see that it can hit zero at some odd step, say $(2k + 1)th$ step. Further $T_0 = 2k + 1$ is same as saying that up to time $2k$, $S_m \geq 1$ and the $(2k+1)th$ step must be -1 (i.e. step down, see figure 1.3). Now we see that the total number of paths joining 1 and 0 by $(2k+1)th$ steps is 2^{2k}.

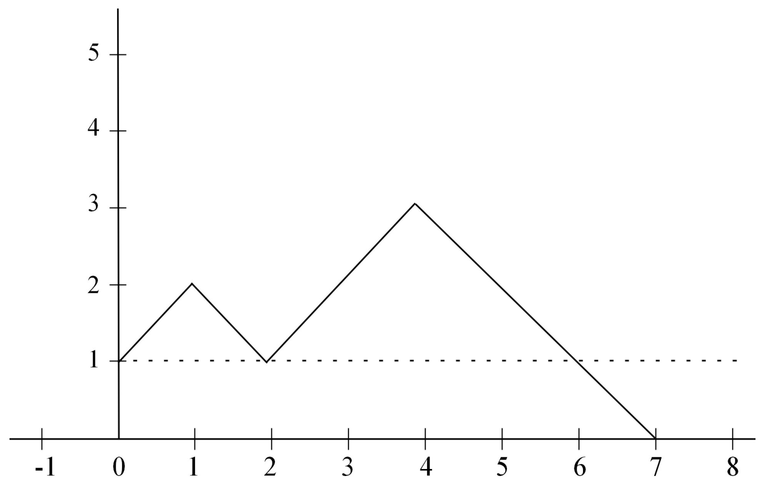


Figure 1.3: *First hitting time at zero.*

And the number of paths for which $S_{2k+1} = 0$ and $S_m > 0$ for $m \leq 2k$ is exactly the number of paths always stay above zero up to time $2k$, starting from 1. It is easy to see that those paths are *Dyck paths* and we know that number of such paths is $C_k = \frac{1}{k+1}\binom{2k}{k}$. Therefore using Stirling's approximation formula we have

$$\begin{aligned}
\mathbb{P}(X = 2k+1) &= \mathbb{P}(T_0 = 2k+1) \\
&= \frac{1}{k+1}\frac{\binom{2k}{k}}{2^{2k}} \\
&\sim \frac{1}{(k+1)2^{2k}}\frac{\sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k}}{2\pi k\left(\frac{k}{e}\right)^{2k}} \\
&\leq \frac{1}{\sqrt{\pi}}k^{-\frac{3}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}(X > t) &= \sum_{k=n}^{\infty}\mathbb{P}(T_0 = 2k+1) \quad \text{(where } n = \lfloor \tfrac{t-1}{2} \rfloor\text{)} \\
&\leq \sum_{k=n}^{\infty}\frac{1}{\sqrt{\pi}}k^{-\frac{3}{2}} \\
&\leq \frac{1}{\sqrt{\pi}}\int_{n-1}^{\infty} x^{-\frac{3}{2}}\,dx \\
&= C_0(n-1)^{-\frac{1}{2}} \quad \text{(where } C_0 = \tfrac{2}{\sqrt{\pi}} \text{ is a constant)} \\
&\leq Ct^{-\frac{1}{2}} \quad \text{(for some larger constant } C\text{)}.
\end{aligned}$$

Hence the proof.

Remark 1.3. We have discussed about translation invariant matching schemes in $\mathbb{Z}$. One can also consider matchings in $\mathbb{Z}^d$ for higher d. The results are similar in $\mathbb{Z}$ and $\mathbb{Z}^2$. For $d \geq 3$, Timár [9] proved that there exists a translation invariant matching scheme in $\mathbb{Z}^d$ such that for any $t > 0$ and any $\epsilon > 0$, we have $\mathbb{P}(X > t) \leq C \exp(-ct^{d-2-\epsilon})$, for some constants $0 < c, C < \infty$.

Chapter 2

Two color matchings in dimensions one and two

In this chapter we consider translation-invariant matching schemes between two independent Poisson point processes $\mathcal{B}$ (say point process of blue points) and $\mathcal{R}$ (say point process of red points) of intensity 1 on $\mathbb{R}^d$ for $d = 1, 2$ and typical matched edge lengths. Before that we need to explain what we mean by a typical edge. For that purpose we modify little bit one of our point processes. For example modify $\mathcal{B}$ by adding one point to $\mathcal{B}$ at the origin and call it $\mathcal{B}^*$ i.e. $\mathcal{B}^* = \mathcal{B} \cup \{0\}$. This $\mathcal{B}^*$ is called *palm version* of $\mathcal{B}$. Now we consider the matched pair length of the point from $\mathcal{B}$ at that origin as a typical matched pair length and denote it by X. We denote the probabilities and expectations of the random variable X by $\mathbb{P}^*$ and $\mathbb{E}^*$ respectively. We consider the questions like for a translation-invariant matching what is the tail behavior of a typical matched pair? How do the answers depend on the dimensions? Answers of these questions were given by A. E. Holroyd, R. Pemantle, Y. Peres and O. Schramm [2] in the following theorems.

Theorem 2.1. *(Upper Bounds): Let $\mathcal{R}$ and $\mathcal{B}$ be two independent Poisson point processes of intensity 1 in $\mathbb{R}^d$. There exist translation-invariant two-color matching schemes satisfying*

i. in $d = 1 : \mathbb{P}^*(X > r) \leq Cr^{-1/2}$, $\forall r > 0$;

ii. in $d = 2 : \mathbb{P}^*(X > r) \leq Cr^{-1}$, $\forall r > 0$.

Here C is a positive constant which depends on the dimension only.

Theorem 2.2. *(Lower Bounds): Let $\mathcal{R}$ and $\mathcal{B}$ be two independent Poisson point processes of intensity 1 in $\mathbb{R}^d$. Then any translation-invariant two-color matching scheme satisfies*

$$\mathbb{E}^* \left[X^{d/2} \right] = \infty, \qquad for\ d = 1, 2.$$

To prove the theorems we need to introduce a few things.

2.1 Partial matching and mass transport

A *partial matching* between two sets U and V is the edge set $\mathcal{E}$ of a bipartite graph $(U, V, \mathcal{E})$ in which every vertex has degree at most one. As before we say $y = \mathcal{E}(x)$ and $x = \mathcal{E}(y)$ if $(x, y) \in \mathcal{E}$ and in addition, we say $\mathcal{E}(x) = \infty$ if x is unmatched, i.e. if x has degree zero. In our case instead of having two deterministic sets we have two point processes $\mathcal{R}$ and $\mathcal{B}$. A *two-color partial matching scheme* $\mathcal{M}$ between two point processes $\mathcal{R}$ and $\mathcal{B}$ is a point process on $[\mathcal{R}] \times [\mathcal{B}]$ which yields almost surely a partial matching between $[\mathcal{R}]$ and $[\mathcal{B}]$ (recall that $[*]$ is the support of the point process $*$).

Lemma 2.3. *(Mass transport principle):*

i. Suppose $t : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty]$ *satisfies* $t(u + w, v + w) = t(u, v)$ *for all* $u, v, w \in \mathbb{Z}^d$, *and write* $t(A, B) := \sum_{u \in A, v \in B} t(u, v)$, *for* $A, B \subset \mathbb{Z}^d$. *Then*

$$t(0, \mathbb{Z}^d) = t(\mathbb{Z}^d, 0).$$

ii. Suppose T *is a random non-negative measure on* $\mathbb{R}^d \times \mathbb{R}^d$ *such that* $T(A, B) := T(A \times B)$ *and* $T(A + w, B + w)$ *are equal in law for all* $w \in \mathbb{Z}^d$ *and* $A, B \subset \mathbb{R}^d$. *Then*

$$\mathbb{E}\left[T(Q, \mathbb{R}^d)\right] = \mathbb{E}\left[T(\mathbb{R}^d, Q)\right] \qquad \textit{(where } Q = [0,1)^d \subset \mathbb{R}^d\textit{)}.$$

Proof. In this lemma the function t or T can be considered as a measure of mass transported from one point to another. The statement (i) is saying that for a translation-invariant mass transport principle, the mass transported from any point is same as mass transported to that point. And the statement (ii) is same about the unit cube Q of $\mathbb{R}^d$. Proofs of these statements are straightforward.
(i) : We see that

$$\begin{aligned} t(0, \mathbb{Z}^d) &= \sum_{u \in \mathbb{Z}^d} t(0, u) \\ &= \sum_{u \in \mathbb{Z}^d} t(-u, 0) \qquad \text{(due to translation-invariance)} \\ &= t(\mathbb{Z}^d, 0). \end{aligned}$$

(ii) : To prove this part define $t(u, v) := [\mathbb{E}T(Q + u, Q + v)]$. Then from the given condition i.e. $T(A, B) \stackrel{d}{=} T(A + w, B + w)$ we can say that

$$\begin{aligned} t(u, v) &= \mathbb{E}\left[T(Q + u, Q + v)\right] \\ &= \mathbb{E}\left[T(Q + u + w, Q + v + w)\right] \qquad \text{(for all } u, v, w \in \mathbb{Z}^d\text{, since } p, q \in \mathbb{Z}^d \text{ implies } p + q \in \mathbb{Z}^d\text{)} \\ &= t(u + w, v + w). \end{aligned}$$

This shows that t satisfies the condition in (i). Then as a corollary of (i) the result follows. $\square$

Now let's say a red point transports a unit mass to its matched partner (which must be a blue point). In this way we can consider a matching as a mass-transportation. The next proposition describes a relation between the intensities of two point processes and partial matching between them.

Proposition 2.4. *(Fairness): Let $\mathcal{R}$ (red points) and $\mathcal{B}$ (blue points) be two independent Poisson point processes of finite intensity, and let $\mathcal{M}$ be a translation-invariant two-color partial matching scheme between $\mathcal{R}$ and $\mathcal{B}$. Then the process of matched red points and the process of matched blue point have same intensity.*

Proof. Let $A \subset [\mathcal{R}]$ and $B \subset [\mathcal{B}]$. Define the following

$$T(A, B) := \#\{x \in A : \mathcal{M}(x) \in B\},$$

i.e. if we consider that each red point sends a unit mass to its partner, then $T(A, B)$ is the total mass transported from the red points in A to their matched pairs in B. Let $Q = [0, 1)^d$, then the intensity of matched red points, i.e. the expected number of matched red points in the unit cube $Q = [0, 1)^d$, is given by the expression

$$\mathbb{E}\left[T(Q, \mathbb{R}^d)\right] = \mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap Q : \mathcal{M}(x) \neq \infty\}\right].$$

Similarly the intensity of the matched blue points is the expected number matched blue of points in Q. But we see that $T(\mathbb{R}^d, Q) = \#\{x \in [\mathcal{R}] \cap \mathbb{R}^d : \mathcal{M}(x) \in Q\}$ which is exactly the number of matched blue points in Q. This implies that the intensity of matched blue points is $\mathbb{E}T(\mathbb{R}^d, Q)$. But we observe that T satisfies the condition of Lemma 2.3(ii). Therefore from Lemma 2.3(ii) we get that

$$\mathbb{E}\left[T(Q, \mathbb{R}^d)\right] = \mathbb{E}\left[T(\mathbb{R}^d, Q)\right].$$

□

We see that a translation-invariant perfect matching is possible only if intensity of matched red and blue points coincide with the intensity of corresponding processes. Therefore as a remark of the previous proposition we can say that translation-invariant perfect matchings are possible only between two processes of same intensity.

2.2 Proof of Theorem 2.1

To prove this we need to give a translation-invariant matching scheme such that

$$\mathbb{P}^*(X > r) \leq C(d) r^{-\frac{d}{2}} \text{ for } d = 1, 2. \tag{2.1}$$

We will produce a sequence of successively coarser random partitions of $\mathbb{R}^d$ in a $\mathbb{Z}^d$ invariant way. Let $\tau_0, \tau_1, \dots$ be uniformly and independently chosen from the set of vertices $\{0,1\}^d$ of the unit d-cube and also chosen independent of the Poisson processes $\mathcal{R}$ and $\mathcal{B}$. We say a set is a $k-box$ if it is of the form

$$[0,2^k)^d + 2^k z + \sum_{i=0}^{k-1} 2^i \tau_i$$

where $z \in \mathbb{Z}^d$. Now we define the matching as follows. Within each 0-box match as many red/blue pairs as possible using some algorithm. For example take the bipartite matching of maximum cardinality which minimizes the total matched length. After 0th step forget the matched pairs and do the same thing for unmatched pairs in 1-boxes. At kth step do the same thing for unmatched pairs within k-boxes. Proceed in this way. In this way we get a translation-invariant matching between two point processes $\mathcal{B}$ and $\mathcal{R}$. Since both of $\mathcal{B}$ and $\mathcal{R}$ have intensity one therefore from the proposition 2.4 we can say that each red point will be matched to some blue point. Now we need to prove (2.1) for this matching. We say a red point is a $k-bad$ point if it is not matched within its k-box by the stage k. Suppose a k-bad point throws a unit mass uniformly to its k-box. Let $A, B \subset \mathbb{R}^d$, now define the following

$$\begin{aligned} T(A,B): &= \sum_{\text{k-bad } x \in A \cap [\mathcal{R}]} 2^{-dk} |\{y \in B : \text{y and x are in same k-box}\}| \\ &= 2^{-dk} \times [\text{Total mass transported to } B \text{ from k-bad points in } A], \end{aligned}$$

where $|R|$=Lebesgue measure of R. Then we see that

$$\begin{aligned} \mathbb{E}\left[T(Q,\mathbb{R}^d)\right] &= \mathbb{E}\left[\#\{\text{k-bad red points in } Q\}\right] \qquad (Q=[0,1)^d) \\ &\geq \mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap Q : |x - \mathcal{M}(x)| > 2^k\sqrt{d}\}\right] \end{aligned}$$

(whence the last inequality follows because diameter of a k-box is $2^k\sqrt{d}$). Let us find out what $\mathbb{E}T(\mathbb{R}^d, Q)$ is. If W is a random k-box that contains Q then we see that

$$\begin{aligned} \mathbb{E}\left[T(\mathbb{R}^d,Q)\right] &= 2^{-dk}\mathbb{E}\left[\#\{\text{k-bad red points in } W\}\right] \\ &= 2^{-dk}\mathbb{E}\left[(\mathcal{R}(W)-\mathcal{B}(W))^+\right] \\ &= 2^{-dk}\mathbb{E}\left[S^+\right]. \end{aligned}$$

where $S := \mathcal{R}[0,2^k)^d - \mathcal{B}[0,2^k)^d$, since W was chosen independent of $\mathcal{R}$ and $\mathcal{B}$ i.e. location of W is independent of $\mathcal{R}$ and $\mathcal{B}$. Now since $\mathcal{R}(W)$ and $\mathcal{B}(W)$ are independent and have Poisson distribution with parameter 2^{dk} therefore the central limit theorem gives us

$$\frac{\mathbb{E}\left[S^+\right]}{\sqrt{2^{dk+1}}} \to \mathbb{E}\left[\chi^+\right] \text{ as } k \to \infty \qquad (\chi \sim N(0,1)). \tag{2.2}$$

Now using Lemma 2.3 we can say that

$$\begin{aligned}
\mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap Q : |x - \mathcal{M}(x)| > 2^k\sqrt{d}\}\right] &\leq \mathbb{E}\left[T(Q, \mathbb{R}^d)\right] \\
&= \mathbb{E}\left[T(\mathbb{R}^d, Q)\right] \\
&= 2^{-dk}\mathbb{E}\left[\#\{\text{k-bad red points in } Q\}\right] \\
&= 2^{-dk}\mathbb{E}\left[S^+\right].
\end{aligned}$$

Therefore using (2.2) we can say that there exists some $C = C(d) \in (0, \infty)$ such that for all $r = 2^k\sqrt{d}$ with $k = 0, 1, \ldots$

$$\mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap Q : |x - \mathcal{M}(x)| > 2^k\sqrt{d}\}\right] \leq Cr^{-\frac{d}{2}}. \tag{2.3}$$

Hence taking $2^{k-1}\sqrt{d} < r < 2^k\sqrt{d}$ the same (2.3) holds (with a modified constant). Since $\mathcal{R}$ and $\mathcal{B}$ are Poisson processes of intensity one,

$$\begin{aligned}
\mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap Q : |x - \mathcal{M}(x)| > r\}\right] &= \mathbb{P}(X > r)\mathbb{E}\left[\{[\mathcal{R}] \cap Q\}\right] \\
&= \mathbb{P}^*(X > r).
\end{aligned}$$

Hence the theorem follows.

2.3 Proof of Theorem 2.2

We will give different proofs for dimension one and two [2].

Dimension 1: For $t > 0$ we have

$$\begin{aligned}
\mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap [0, 2t] : \mathcal{M}(x) \notin [0, 2t]\}\right] &\leq \mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap [0, 2t] : |\mathcal{M}(x) - x| > x \wedge (2t - x)\}\right] \\
&= \int_0^{2t} \mathbb{P}^*[X > x \wedge (2t - x)]\, dx \\
&= 2\int_0^{t} \mathbb{P}^*[X > x]\, dx.
\end{aligned}$$

Now the central limit theorem gives

$$\mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap [0, 2t] : \mathcal{M}(x) \notin [0, 2t]\}\right] \geq \mathbb{E}\left[(\mathcal{R}[0, 2t] - \mathcal{B}[0, 2t])^+\right] \sim Ct^{\frac{1}{2}},$$

as $t \to \infty$, for some $C \in (0, \infty)$. Therefore

$$t^{-\frac{1}{2}} \int_0^t \mathbb{P}^*[X > x]\, dx \geq C \quad \text{as } t \to \infty. \tag{2.4}$$

Now if $\mathbb{E}^*\left[X^{\frac{1}{2}}\right] < \infty$ then the dominated convergence theorem gives

$$t^{-\frac{1}{2}} \int_0^t \mathbb{P}^*[X > x]\, dx \to 0 \ \text{ as } t \to \infty,$$

which contradicts (2.4). Hence the proof.

Dimension 2: One can intuitively see that if $\mathbb{E}^*[X] = \infty$ then the number of intersections of matched edges with the unit square Q will be infinite and it will be finite if $\mathbb{E}^*[X] < \infty$. We will show $\mathbb{E}^* X = \infty$ using this idea. Let us define the line segment $\langle x, y\rangle := \{\lambda x + (1-\lambda)y : \lambda \in [0,1]\} \subset \mathbb{R}^d$. We use the following lemma to prove our theorem.

Lemma 2.5. *(Edge intersections): For any translation-invariant 1-color or 2-color matching scheme $\mathcal{M}$ (of any translation-invariant point processes) which satisfies $\mathbb{E}^*[X] < \infty$, the number of matching edges $\{x, y\} \in [\mathcal{M}]$ such that the line segment $\langle x, y\rangle$ intersects the unit cube Q has finite expectation.*

Proof. To prove this, we will once again use the mass transport principle. Consider the mass transport function

$$t(u, v) := \mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap (Q+u) : \langle x, \mathcal{M}(x)\rangle \text{ intersects } Q+v\}\right],$$

where $u, v \in \mathbb{Z}^d$. Since an edge of length l intersects at most $d(1+l)$ cubes of the form $z+Q$, where $z \in \mathbb{Z}^d$, we have

$$\begin{aligned} t(0, \mathbb{Z}^d) &= \mathbb{E}\left[\#\{x \in [\mathcal{R}] \cap Q\}\mathbb{E}\#\{v \in \mathbb{Z}^d : \langle x, \mathcal{M}(x)\rangle \text{intersects } Q+v\}\right] \\ &\leq d + d\mathbb{E}^*[X] \quad \text{(Since } \mathcal{R} \text{ has intensity 1)} \\ &< \infty. \end{aligned}$$

Hence by Lemma 2.3 we get

$$\mathbb{E}\#\left[\{\text{matching edges intersecting } Q\}\right] = t(\mathbb{Z}^d, 0) < \infty$$

□

Proof of the original Theorem 2.2 in dimension two: Without loss of generality we may assume that the matching scheme $\mathcal{M}$ is ergodic under translations of $\mathbb{R}^2$; if not we apply the claimed result to the ergodic components. Suppose that an ergodic matching scheme $\mathcal{M}$ satisfies $\mathbb{E}^*[X] < \infty$. Now for two distinct points $x, y \in \mathbb{R}^2$ we define the random variable $K(x, y)$ to be the number of matched edges which intersect the directed line segment $\langle x, y\rangle$ with the red point (i.e. point of $[\mathcal{R}]$) on the left and the blue point (i.e. point of $[\mathcal{B}]$) on the right. The Lemma 2.5 together with the assumption $\mathbb{E}^*[X] < \infty$ will imply that $\mathbb{E}[K(x, y)] < \infty$ for any fixed x, y. Also note one property (additive) of $K(x, y)$ is if $z \in \langle x, y\rangle$, then $K(x, z) + K(z, y) = K(x, y)$. Fix a unit vector $u \in \mathbb{R}^2$. Using the ergodic theorem and ergodicity of $\mathcal{M}$ we can say that

$$\frac{K(0, nu)}{n} \xrightarrow{L^1} \mathbb{E}\left[K(0, u)\right] \text{ as } n \to \infty,$$

Since $K(v, v+nu)$ has the same distribution as $K(0, nu)$, it follows that

$$\frac{K(v_n, v_n + nu)}{n} \xrightarrow{L^1} \mathbb{E}\left[K(0,u)\right] \quad \text{as } n \to \infty, \tag{2.5}$$

for any deterministic sequence $v_n \in \mathbb{R}^2$. Now let us denote $S_n := [0,n]^2 \subset \mathbb{R}^2$ and the vertices $s_1^n = (0,0); s_2^n = (n,0); s_3^n = (n,n); s_4^n = (0,n)$. We observe that $\mathcal{R}(S_n) - \mathcal{B}(S_n)$ is exactly equal to the number of red points inside of S_n matched with blue points outside of S_n minus the number of blue points inside matched with red points outside. Hence,

$$\mathcal{R}(S_n) - \mathcal{B}(S_n) = K_+ - K_-, \tag{2.6}$$

where

$$\begin{aligned} K_+ : &= K(s_1^n, s_2^n) + K(s_2^n, s_3^n) + K(s_3^n, s_4^n) + K(s_4^n, s_1^n) \\ K_- : &= K(s_1^n, s_4^n) + K(s_4^n, s_3^n) + K(s_3^n, s_2^n) + K(s_2^n, s_1^n). \end{aligned}$$

(note: The edges which intersect two sides of S_n will contribute nothing because the contribution of that edge to K_+ cancels with that to K_-). Let $u = (1,0)$ and $v = (0,1)$, then (2.5) implies

$$\frac{K_+}{n} \xrightarrow{L^1} \mathbb{E}\left[K(0,u)\right] + \mathbb{E}\left[K(0,v)\right] + \mathbb{E}\left[K(0,-u)\right] + \mathbb{E}\left[K(0,-v)\right]$$

and

$$\frac{K_-}{n} \xrightarrow{L^1} \mathbb{E}\left[K(0,v)\right] + \mathbb{E}\left[K(0,u)\right] + \mathbb{E}\left[K(0,-v)\right] + \mathbb{E}\left[K(0,-u)\right].$$

Hence $\frac{K_+ - K_-}{n} \xrightarrow{L^1} 0$; i.e.

$$\mathbb{E}\left[|K_+ - K_-|\right] = o(n). \tag{2.7}$$

On the other hand the central limit theorem gives $\frac{\mathcal{R}(S_n) - \mathcal{B}(S_n)}{n\sqrt{2}} \xrightarrow{d} N(0,1)$ which is not compatible with (2.7). Therefore $\mathbb{E}^*[X] < \infty$ is not true.

Chapter 3

Two color matchings in dimensions more than two

In the previous chapter we discussed the results for two-color matching in dimensions one and two. We observed that the tail behavior of a matched pair length are similar for dimension one and two. However the things are drastically changed in dimension more than two! For dimensions one and two, even $\mathbb{E}^*\left[X^{d/2}\right]$ does not exist, whereas in dimension three or more than three the exponential moments exist. Following results are due to A. E. Holroyd, R. Pemantle, Y. Peres and O. Schramm [2] about the tail behavior of a matched pair length in dimension more than two.

Theorem 3.1. *(Lower Bounds): Let $\mathcal{R}$ and $\mathcal{B}$ be two independent Poisson point processes in $\mathbb{R}^d$ $(d \geq 3)$ of intensity one. Then there exists a positive constant C such that for any translation-invariant matching scheme*

$$\mathbb{E}^*\left[e^{CX^d}\right] = \infty,$$

where the positive constant C depends only on the dimension d.

The upper bounds regarding this is

Theorem 3.2. *(Upper Bounds): Let $\mathcal{R}$ and $\mathcal{B}$ be two independent Poisson point processes in $\mathbb{R}^d$ $(d \geq 3)$ of intensity one. There exists a positive constant c translation-invariant two-color matching schemes satisfying*

$$\mathbb{E}^*\left[e^{cX^d}\right] < \infty,$$

where the positive constant c depends only on the dimension d.

Therefore from the previous two theorems we see that in some sense these are the sharp bounds for a matched pair length. It is easy to see the lower bound.

3.1 Proof of Theorem 3.1

We observe that the distance between the blue point at the origin and its matched partner is at least the minimum distance between the origin and all other red points. Therefore if we denote X as the matched pair length for the blue point at the origin, we see that

$$\begin{aligned} \mathbb{P}^*(X > r) &= \text{Probability that there is no red point in } B(0,r) \\ &= e^{-C(d)r^d}, \quad \text{where } C(d) = \text{volume of the unit ball } B(0,1). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}^*\left[e^{CX^d}\right] &= \int_{r=1}^{\infty} \mathbb{P}^*\left(e^{CX^d} > r\right)\, dr \\ &= \int_{r=1}^{\infty} \mathbb{P}^*\left(CX^d > \log r\right)\, dr \\ &= \int_{r=1}^{\infty} \mathbb{P}^*\left(X > \left(\frac{\log r}{C}\right)^{1/d}\right)\, dr \\ &= \int_{r=1}^{\infty} e^{-\frac{C(d)}{C}\log r}\, dr \\ &= \int_{r=1}^{\infty} r^{-\frac{C(d)}{C}}\, dr. \end{aligned}$$

Observe that if we take $C = C(d)$ then the above integration blows up. i.e. $\mathbb{E}^* e^{C(d)X^d} = \infty$.

3.2 Proof of Theorem 3.2

To prove this we will use a result proved by M. Talagrand [6, eqn. (1.8)]. The result says that in $d \geq 3$ if $\mathcal{R}_n$ and $\mathcal{B}_n$ are uniformly independently distributed in $[0,1]^d$ consists of n points each, then for each n there exists a two color matching scheme $\mathcal{F}_n$ between $\mathcal{R}_n$ and $\mathcal{B}_n$ such that

$$\mathbb{P}(G_n) \geq 1 - \frac{1}{n^2}, \tag{3.1}$$

where

$$G_n := \left\{ \frac{1}{n} \sum_{x \in \mathcal{R}_n} \exp(cn|x - \mathcal{F}_n(x)|^d) \leq 2 \right\}.$$

Here C is a constant that depends on the dimension d but not on n. We want to construct a matching $\mathcal{M}$ between $\mathcal{R}$ and $\mathcal{B}$. Let $\widetilde{\mathcal{F}_n}$ be $\mathcal{F}_n$ conditioned on the event G_n. We construct a translation-invariant matching scheme $\mathcal{M}_n$ by scaling $\widetilde{\mathcal{F}_n}$ to a unit cube of volume n and tiling the whole $\mathbb{R}^d$ with identical copies of this matching, with origin chosen uniformly at random. In other words, regarding a two color matching scheme $\mathcal{M}$ as a point process of ordered pairs

$(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ (i.e. presence of $(r,b)\in[\mathcal{M}]$ means that $r\in[\mathcal{R}]$, $b\in[\mathcal{B}]$, and r is matched to b), define

$$\mathcal{M}_n(A\times B):=\sum_{z\in\mathbb{Z}^d}\widetilde{\mathcal{F}_n}\left(n^{\frac{1}{d}}(A+U+z)\times n^{\frac{1}{d}}(B+U+z)\right),$$

where U is uniformly distributed on $[0,1]^d$ and independent of $\widetilde{\mathcal{F}_n}$. Then (3.1) gives us

$$\mathbb{E}\left[\int\int\exp(c|x-y|^d)\mathbf{1}_{x\in A}\mathcal{M}_n(dx\times dy)\right]\leq 2(\text{Volume of } A), \tag{3.2}$$

for any borel $A\subset\mathbb{R}^d$. Now by (3.2) the sequence $\{\mathcal{M}_n\}_n$ is tight in the vague topology of measures on $\mathbb{R}^d\times\mathbb{R}^d$. Therefore there exists a subsequence of $\{\mathcal{M}_n\}_n$ that converges vaguely to a some point process $\mathcal{M}$, a two-color matching scheme between marginal point processes $\mathcal{M}(.,\mathbb{R}^d)$ and $\mathcal{M}(\mathbb{R}^d,.)$. These processes are independent Poisson point processes of intensity 1. The process $\mathcal{M}$ inherits the translation-invariance of $\mathcal{M}_n$ and it satisfies (3.2) (with $\mathcal{M}_n$ replaced by $\mathcal{M}$), which implies $\mathbb{E}^*\left[e^{cX^d}\right]\leq 2$.

Chapter 4

On optimal matching

Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ be a set of i.i.d. uniform points in $[0,1]^2$. Given $\{X_i\}$ and $\{Y_i\}$, there are $n!$ possible matchings between them. Our main goal is to analyze the minimum total matched edge length. Define

$$T_p := \inf_{\pi \in \mathcal{P}(n)} \sum_{i=1}^{n} \|X_i - Y_{\pi(i)}\|^p, \quad 1 \le p < \infty,$$

where $\mathcal{P}(n)$ is the set of all permutations of the numbers $1, \ldots, n$. Note that T_p also depends on n. To avoid notational complication, we are not writing that. The main result about T_1 is due to Ajtai, Komlos, and Tusnády [1], which says that there exists constants L_1 and L_2 such that

$$L_1 \sqrt{n \log n} < T_1 < L_2 \sqrt{n \log n} \quad \text{with probability } 1 - o(1). \tag{4.1}$$

This theorem is also saying that the average optimal matched edge length is like $\sqrt{\frac{\log n}{n}}$. For calculational conveniences we take all logarithms in this chapter of base 2. From here onwards L will denote a constant and may change value from line to line unless otherwise stated.

Instead of looking at $[0,1]^2$, if we look at $[0,1]^d$ for $d \ge 3$, then we can get rid of the extra $\log n$ factor in the average matched edge length and it will be just $n^{-\frac{1}{d}}$. We will not discuss it here.

4.1 Concentration inequality of T_1:

Lemma 4.1. *We have the following concentration inequality of T_1 around its mean*

$$\mathbb{P}\left(|T_1 - \mathbb{E}[T_1]| > 4\sqrt{2n \log n}\right) \le \frac{2}{n^2}.$$

Proof. Let us define $\mathcal{F}_k$ be the σ-algebra generated by $\{X_1, \ldots, X_k, Y_1, \ldots, Y_k\}$, and $\mathcal{F}_0$ be the trivial σ-algebra. Then observe that $M_k := \mathbb{E}[T_1 | \mathcal{F}_k]$ is a martingale sequence for $0 \le k \le n$, and we have

$$T_1 - \mathbb{E}[T_1] \quad = \quad \sum_{k=1}^{n} \{M_k - M_{k-1}\}.$$

We claim that $|M_k - M_{k-1}|$ is uniformly bounded by $2\sqrt{2}$ for all $1 \le k \le n$.

Proof of $|M_k - M_{k-1}| \le 2\sqrt{2}$***:*** We see that T_1 is actually a function of $X_1, \dots, X_n, Y_1, \dots, Y_n$. Fix an $k \in \{1, \dots, n\}$, and let us take $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $x' = (x'_1, \dots, x'_n)$, $y' = (y'_1, \dots, y'_n)$, where $x_j = x'_j$ and $y_j = y'_j$ for all $j \neq k$. Let π be the permutation for which $T_1(x, y) = \sum_{i=1}^{n} \|x_i - y_{\pi(i)}\|$. Let us define $\hat{T}_1(x', y') := \sum_{i=1}^{n} \|x'_i - y'_{\pi(i)}\|$. Note that (x, y) and (x', y') differs only at x'_k and y'_k. But $\|x'_k - y'_{\pi(k)}\| \le \sqrt{2}$, therefore $\hat{T}_1(x', y') - T_1(x, y) \le 2\sqrt{2}$. But $T_1(x', y') \le \hat{T}_1(x', y')$, since $\hat{T}_1$ corresponds to a particular matching and T_1 is infimum over all matchings. So, we have $T_1(x', y') - T_1(x, y) \le 2\sqrt{2}$. Similarly $T_1(x, y) - T_1(x', y') \le 2\sqrt{2}$. Hence

$$|T_1(x, y) - T_1(x', y')| \le 2\sqrt{2}. \tag{4.2}$$

Now let us take $1 \le k \le n$, $X_1 = x_1, \dots, X_k = x_k$, $Y_1 = y_1, \dots, Y_k = y_k$, and two i.i.d. uniform random variables U, V in $[0, 1]^2$. Using (4.2) we get that

$$\begin{aligned} &|T_1(x_1, \dots, x_k, X_{k+1}, \dots, X_n, y_1, \dots, y_k, Y_{k+1}, \dots, Y_n)| \\ -T_1(x_1, \dots, x_{k-1}, U, X_{k+1}, \dots, X_n, y_1, \dots, y_{k-1}, V, Y_{k+1}, \dots, Y_n)| &\le 2\sqrt{2}, \end{aligned}$$

which gives us $|[T_1|\mathcal{F}_k] - [T_1|\mathcal{F}_{k-1}]| \le 2\sqrt{2}$. Hence taking expectation we get our result. □

Now by Hoeffding's inequality we have $\mathbb{P}(T_1 - \mathbb{E}[T_1] > t) \le e^{-t^2/16n}$. Taking $t = 4\sqrt{2n \log n}$ we get that

$$\begin{aligned} \mathbb{P}(|T_1 - \mathbb{E}[T_1]| > t) &\le 2e^{-\frac{t^2}{16n}} \\ &= \frac{2}{n^2}. \end{aligned} \tag{4.3}$$

□

This gives us a concentration inequality of T_1 around $\mathbb{E}[T_1]$. Therefore if we can prove that $\mathbb{E}[T_1] = \Theta(\sqrt{n \log n})$, then (4.1) will follow from there.

In this chapter we will only prove $\mathbb{E}[T_1] = O(\sqrt{n \log n})$. The next theorem is a key ingredient of that and it states more than $\mathbb{E}[T_1] = O(\sqrt{n \log n})$.

4.2 Bound on $\mathbb{E}[T_1]$:

Theorem 4.2 (Talagrand, M. and Yukich, JE). *[8] Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ be a set of i.i.d. uniform points in $[0,1]^2$. Then there exists a constant L and events C_n such that*

$$\mathbb{E}_{C_n}\left[\inf_{\pi \in \mathcal{P}(n)} \frac{1}{n} \sum_{i=1}^{n} \exp\left(\frac{n\|X_i - Y_{\pi(i)}\|^2}{L \log n}\right)\right] \leq L,$$

where $\mathcal{P}(n)$ is the set of all permutations of the numbers $1, \ldots, n$, $\mathbb{P}(C_n) \geq 1 - \frac{1}{n^2}$, and $\mathbb{E}_A[X] = \int_A X \, dP$.

We see that any $\pi \in \mathcal{P}(n)$ induces a matching between $\{X_i\}$ and $\{Y_i\}$. Sometimes we call $\pi \in \mathcal{P}(n)$ as a matching between $\{X_i\}$ and $\{Y_i\}$.

Consequences of the Theorem 4.2:

1. By convexity of e^x we get

$$\exp\left[\frac{1}{n}\sum_{i=1}^{n} \frac{n\|X_i - Y_{\pi(i)}\|^2}{L \log n}\right] \leq \frac{1}{n}\sum_{i=1}^{n} \exp\left(\frac{n\|X_i - Y_{\pi(i)}\|^2}{L \log n}\right),$$

which implies that

$$\mathbb{E}\left[\exp\left(\frac{T_2}{L \log n}\right) 1_{C_n}\right] \leq L.$$

So we get

$$\mathbb{P}\left(\left(\frac{T_2}{L \log n}\right) 1_{C_n} > t\right) = \mathbb{P}\left(\exp\left(\frac{T_2}{L \log n}\right) 1_{C_n} > e^t\right) \leq \frac{L}{e^t},$$

i.e. $\frac{T_2}{\log n}$ has an exponential tail on the set C_n.

2. From the definition of T_p, we have $T_1 = \inf_{\pi \in \mathcal{P}(n)} \sum_{i=1}^{n} \|X_i - Y_{\pi(i)}\|$. Now by the convexity of the function x^2 we get that for any permutation $\pi \in \mathcal{P}(n)$

$$\left(\frac{1}{n}\sum_{i=1}^{n} \|X_i - Y_{\pi(i)}\|\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n} \|X_i - Y_{\pi(i)}\|^2$$

On the right hand side taking the permutation π^* which gives us $T_2 = \sum_{i=1}^{n} \|X_i - Y_{\pi^*(i)}\|^2$

we get that

$$\begin{aligned} T_1^2 &= \left(\inf_{\pi \in \mathcal{P}(n)} \sum_{i=1}^{n} \|X_i - Y_{\pi(i)}\| \right)^2 \\ &\leq \left(\sum_{i=1}^{n} \|X_i - Y_{\pi^*(i)}\| \right)^2 \\ &\leq n \sum_{i=1}^{n} \|X_i - Y_{\pi^*(i)}\|^2 \\ &= nT_2. \end{aligned}$$

Therefore $\mathbb{E}\left[\exp\left(\frac{T_1^2}{Ln\log n}\right) 1_{C_n}\right] \leq L$. Now using the convexity of the functions x^2 and e^x we get

$$\begin{aligned} \left(\mathbb{E}\left[\frac{T_1}{\sqrt{Ln\log n}} \right] \right)^2 &\leq \mathbb{E}\left[\frac{T_1^2}{Ln\log n} \right] \\ &= \log\left(\exp\left[\mathbb{E}\left(\frac{T_1^2}{Ln\log n} \right) \right] \right) \\ &\leq \log\left(\mathbb{E}\left[\exp\left(\frac{T_1^2}{Ln\log n} \right) \right] \right). \end{aligned}$$

Hence using Theorem 4.2 we get

$$\begin{aligned} \mathbb{E}[T_1] &= \mathbb{E}[T_1 1_{C_n}] + \mathbb{E}[T_1 1_{C_n^c}] \\ &\leq L'\sqrt{n\log n} + \frac{n\sqrt{2}}{2n^2} \\ &\leq L\sqrt{n\log n}, \end{aligned} \tag{4.4}$$

Which gives us $\mathbb{E}[T_1] = O(\sqrt{n\log n})$.

Remark 4.3. From P. Shor's thesis [4] we have for any $\epsilon > 0$

$$T_1 \geq L_1\sqrt{n\log n} \qquad \text{with probability } 1 - 2^{-n^\epsilon}, \tag{4.5}$$

which gives us $\mathbb{E}[T_1] = \Omega(\sqrt{n\log n})$.

Proof of the Theorem 4.2: This proof is due to Talagrand and Yukich [8]. In this proof we will use a beautiful technique used by Ajtai, Komlos, Tusandy [1]; P. Shor [4].

Imagine that the interior of $[0,1]^2$ is made of an elastic membrane and the points $X_1, \ldots, X_n$; $Y_1, \ldots, Y_n$ are placed on that membrane. First divide the whole square horizontally into two equal parts. Now shift the divider up/down so that the area of each half becomes proportional to the number of points from $\{X_1, \ldots, X_n\}$ inside it. At the second step divide each half vertically into two equal parts and move the dividers left/right to obtain

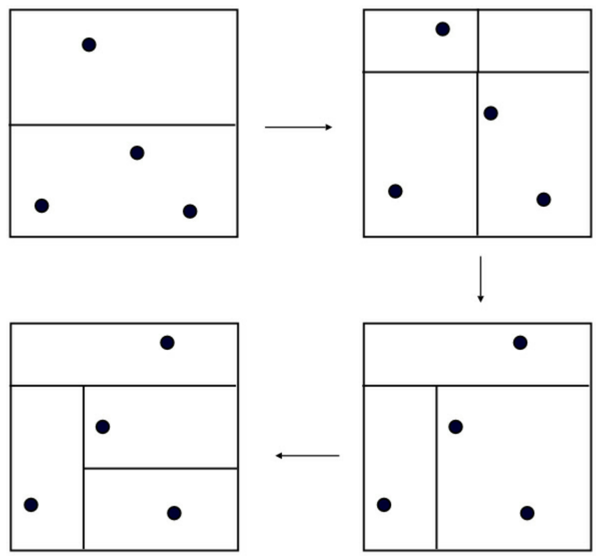


Figure 4.1: *Shifting the points*

rectangles of areas proportional to the number of points (from $\{X_1, \ldots, X_n\}$) inside it. At the third step divide each four rectangles horizontally into two equal parts and do the same kind of shifting business for each rectangle as we did in the first step. Do the same thing repeatedly until we get rectangles containing only one point inside it. So, in odd steps we divide the existing rectangles horizontally and in even steps we divide vertically (figure 4.1).

Note that since the interior of the unit square $[0,1]^2$ is made of an elastic membrane, so during shifting dividers the points in it will move accordingly, namely the points near the divider will move more than the points away from the divider. After doing this process sufficiently many times we will get that each rectangle contains one transformed sample point. Note that some rectangles will vanish if at some step it does not contain any point. Also note that since at each step the area of the rectangles are proportional to the number points inside it, therefore at the final step each non degenerate rectangle will have area $\frac{1}{n}$. We also do the same kind of shifting business for $Y_1, \ldots, Y_n$.

Now let after transformation $\hat{X}_i$'s and $\hat{Y}_i$'s be the final positions of X_i's and Y_i's respectively, and $\hat{R}_i$, $\hat{B}_i$ be the rectangles which contains $\hat{X}_i$, $\hat{Y}_i$ respectively. Also let $\tilde{X}_i$'s and $\tilde{Y}_i$'s be the positions of X_i's and Y_i's at $2r$th step, and $\tilde{R}_i$, $\tilde{B}_i$ are the rectangles which contains $\tilde{X}_i$ and $\tilde{Y}_i$ respectively at $2r$th step, where $2^{-r} = 10\left(\frac{\log n}{n}\right)^{\frac{1}{2}}$. Practically for doing the calculations we will consider the first $2r$ steps only.

Now we will find two sets A_n, B_n and a constant L such that $\mathbb{P}(A_n^c) \leq \frac{1}{2n^2}$, $\mathbb{P}(B_n^c) \leq \frac{1}{2n^2}$,

$$\mathbb{E}_{A_n}\left[\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{n\|X_i - \tilde{X}_i\|^2}{L\log n}\right)\right] \leq L$$

$$\mathbb{E}_{B_n}\left[\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{n\|Y_i-\tilde{Y}_i\|^2}{L\log n}\right)\right]\le L \tag{4.6}$$

and

$$\mathbb{E}_{A_n\cap B_n}\left[\inf_{\pi\in\mathcal{P}(n)}\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{n\|\tilde{X}_i-\tilde{Y}_{\pi(i)}\|^2}{L\log n}\right)\right]\le L \tag{4.7}$$

The theorem will follow from these results because for any permutation $\pi\in\mathcal{P}(n)$, by the convexity of x^2 we have

$$\begin{aligned}\left\|\frac{X_i-Y_{\pi(i)}}{3}\right\|^2 &= \left\|\frac{X_i-\tilde{X}_i}{3}+\frac{\tilde{X}_i-\tilde{Y}_{\pi(i)}}{3}+\frac{\tilde{Y}_{\pi(i)}-Y_{\pi(i)}}{3}\right\|^2\\ &\le \frac{1}{3}\left[\|X_i-\tilde{X}_i\|^2+\|\tilde{X}_i-\tilde{Y}_{\pi(i)}\|^2+\|\tilde{Y}_{\pi(i)}-Y_{\pi(i)}\|^2\right].\end{aligned}$$

Therefore for any constant $L>0$ and $n>1$

$$\begin{aligned}\frac{n\|X_i-Y_{\pi(i)}\|^2}{L\log n} &\le \frac{3n}{L\log n}\left[\|X_i-\tilde{X}_i\|^2+\|\tilde{X}_i-\tilde{Y}_{\pi(i)}\|^2+\|\tilde{Y}_{\pi(i)}-Y_{\pi(i)}\|^2\right]\\ \Rightarrow\quad & \frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{n\|X_i-Y_{\pi(i)}\|^2}{L\log n}\right)\\ &\le \frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{3n}{L\log n}\left\{\|X_i-\tilde{X}_i\|^2+\|\tilde{X}_i-\tilde{Y}_{\pi(i)}\|^2+\|\tilde{Y}_{\pi(i)}-Y_{\pi(i)}\|\right\}\right)\\ &\le \frac{1}{3n}\sum_{i=1}^{n}\exp\left(\frac{9n\|X_i-\tilde{X}_i\|^2}{L\log n}\right)+\frac{1}{3n}\sum_{i=1}^{n}\exp\left(\frac{9n\|\tilde{X}_i-\tilde{Y}_{\pi(i)}\|^2}{L\log n}\right)\\ &\quad+\frac{1}{3n}\sum_{i=1}^{n}\exp\left(\frac{9n\|\tilde{Y}_{\pi(i)}-Y_{\pi(i)}\|^2}{L\log n}\right),\end{aligned}$$

using the convexity of e^x in the last inequality. Now by taking the minimum over all $\pi\in\mathcal{P}(n)$ and expectation over $C_n:=A_n\cap B_n$ we get

$$\mathbb{E}_{C_n}\left[\inf_{\pi\in\mathcal{P}(n)}\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{n\|X_i-Y_{\pi(i)}\|^2}{L\log n}\right)\right]\le L.$$

Now to complete the proof of the theorem we need to prove the equations (4.6) and (4.7).

Proof of (4.6). First we need to define A_n, B_n and show that $\mathbb{P}(A_n^c)\le\frac{1}{2n^2}$, $\mathbb{P}(B_n^c)\le\frac{1}{2n^2}$. We can divide the whole unit square into 2^{2l} equal sub squares just by dividing each side of $[0,1]^2$ into 2^l equal parts. Let us define for each $0\le l\le r$, $\mathscr{S}_l:=\{S_1,\dots,S_{2^{2l}}\}$ be the set of 2^{2l} dyadic squares. But after $2l$ number of steps those squares will be transformed into rectangles. Let $\mathcal{S}_l:=\{\tilde{S}_1,\dots,\tilde{S}_{2^{2l}}\}$ be the set of all transformed dyadic squares after $2l^{th}$

step. Take any $S \in \mathscr{S}_l$, and let $\tilde{S} \in \mathcal{S}_l$ be the transformed version of it. Since the sample points are also being moved during the transformation, therefore the number of sample points in S is same as the number of transformed sample points in $\tilde{S}$. Take an $S \in \mathscr{S}_l$ divide it horizontally into two equal parts, and let us say H be the upper half of it. Define

$$p_{l,S} = \frac{\text{card}\{i : X_i \in H\}}{\text{card}\{i : X_i \in S\}}.$$

Now divide each of H and $S\backslash H$ vertically into two equal parts, and let us say H^* and H^{c*} be the left halves of H and $S\backslash H$ respectively. Define

$$p^*_{l,H} = \frac{\text{card}\{i : X_i \in H^*\}}{\text{card}\{i : X_i \in H\}}$$

and

$$p^*_{l,H^c} = \frac{\text{card}\{i : X_i \in H^{c*}\}}{\text{card}\{i : X_i \in S\backslash H\}}.$$

Now define, $A_n \subset \Omega$ be the set of $\omega \in \Omega$ such that for all $1 \le l \le r$
(i) $\left|p_{l,S} - \frac{1}{2}\right| \le L\left(\frac{l}{n}\right)^{\frac{1}{2}} 2^{\frac{r+l}{2}}$
(ii) $\left|p^*_{l,H} - \frac{1}{2}\right| \le L\left(\frac{l}{n}\right)^{\frac{1}{2}} 2^{\frac{r+l}{2}}$
(iii) $\left|p^*_{l,H^c} - \frac{1}{2}\right| \le L\left(\frac{l}{n}\right)^{\frac{1}{2}} 2^{\frac{r+l}{2}}$,
where L is a constant and we will say its value later.

Lemma 4.4. *For n large enough, $\mathbb{P}(A_n^c) \le \frac{1}{2n^2}$.*

Proof. Fix an $S \in \mathscr{S}_l$ and define M:= number of sample points in S and X:= number of sample points in the upper half H of S. Then we see that $M \sim \text{Bin}(n, 2^{-2l})$ and $X|M \sim \text{Bin}(M, \frac{1}{2})$. Therefore using Hoeffding's inequality we get

$$\begin{aligned}
\mathbb{P}\left(\left|p_{l,\tilde{S}} - \frac{1}{2}\right| > t\right) &= \mathbb{E}\left[\mathbb{P}\left(\left|p_{l,\tilde{S}} - \frac{1}{2}\right| > t \,\Big|\, M\right)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\left|X - \frac{M}{2}\right| > Mt \,\Big|\, M\right)\right] \\
&\le 2\mathbb{E}\left[e^{-2Mt^2}\right] \\
&= 2\left[q_l + p_l e^{-2t^2}\right]^n, \qquad \text{where } p_l = 2^{-2l} \text{ and } q_l = 1 - p_l \\
&= 2\left[1 - p_l\left(1 - e^{-2t^2}\right)\right]^n \\
&\le 2e^{-np_l\left(1 - e^{-2t^2}\right)}.
\end{aligned}$$

Now recalling that $2^r = \frac{1}{10}\left(\frac{n}{\log n}\right)^{\frac{1}{2}}$, we observe $r \le \log n$. On the other hand using the mean value property and decreasing property of the function e^{-x} we have $1 - e^{-2t^2} \ge 2t^2 e^{-2t^2}$. Now in our case $t = L\left(\frac{l}{n}\right)^{\frac{1}{2}} 2^{\frac{r+l}{2}}$ (t depends on l). Therefore,

$$\begin{aligned}
2^{2l}\frac{L^2 l}{n 2^l} 2^r = \; & t^2 \; \leq \frac{L^2 r}{n} 2^{2r} && \text{[since } 1 \leq l \leq r\text{]} \\
\Rightarrow \quad 2^{2l}\frac{L^2 r}{n 2^r} 2^r \leq \; & t^2 \; \leq \frac{L^2 \log n}{n} \frac{n}{100 \log n} && \text{[since } x2^{-x} \text{ is a decreasing function for } x \geq 1\text{]} \\
\Rightarrow \quad 2^{2l}\frac{L^2 r}{n} \leq \; & t^2 \; \leq \frac{L^2}{100}.
\end{aligned}$$

Now we have the following estimate

$$\begin{aligned}
p_l\left(1 - e^{-2t^2}\right) & \geq 2^{-2l} 2t^2 e^{-2t^2} \\
& \geq \frac{2L^2 r}{n} e^{-\frac{L^2}{50}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
e^{-np_l\left(1 - e^{-\frac{t^2}{2}}\right)} & \leq e^{-cr} \qquad \text{[where } c = 2L^2 e^{-\frac{L^2}{50}}\text{]} \\
& = 2^{-cr \log e}.
\end{aligned}$$

Now using the above estimates we get that

$$\begin{aligned}
\mathbb{P}\left(\exists\, \tilde{S} \in \mathcal{S}_l \text{ s.t. } \left|p_{l,\tilde{S}} - \frac{1}{2}\right| > L\left(\frac{l}{n}\right)^{\frac{1}{2}} 2^{\frac{r+l}{2}}\right) & \leq 2^{2l+1-cr\log e} \\
& \leq 2^{2r+1-cr\log e}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{P}\left(\exists\, \tilde{S} \in \mathcal{S}_l \text{ and } H \text{ s.t. } \left|p^*_{l,H} - \frac{1}{2}\right| > L\left(\frac{l}{n}\right)^{\frac{1}{2}} 2^{\frac{r+l}{2}} \text{or} \left|p^*_{l,H^c} - \frac{1}{2}\right| > L\left(\frac{l}{n}\right)^{\frac{1}{2}} 2^{\frac{r+l}{2}}\right) \\
\leq 2^{2r+2-\frac{cr}{2}\log e}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}(A_n^c) \leq r\left(2^{2r+1-cr\log e} + 2^{2r+2-\frac{cr}{2}\log e}\right) & \leq r2^{2r+3-\frac{cr}{2}\log e} \\
& \leq 8\left(\frac{\sqrt{10}\log n}{n}\right)^{\frac{c}{4}\log e - 1} \log n.
\end{aligned}$$

Now taking $L = \frac{5}{2}$ we have $c = \frac{25}{2}e^{-\frac{1}{8}}$. In that case $\frac{c}{4}\log e - 1 > 2.97$, and we have

$$\begin{aligned}
\mathbb{P}(A_n^c) & \leq 8\left(\frac{\sqrt{10}\log n}{n}\right)^{2.97} \log n \\
& \leq \frac{1}{2n^2} \qquad \text{for sufficiently large } n.
\end{aligned}$$

$\square$

In all these transformations one thing we need to worry about is that how do the dyadic squares look like after the transformations. As we discussed before that some squares will vanish during the transformations. We call them as *degenerate* squares. We are interested only on non degenerate transformed squares. Define the aspect ratio of a rectangle as the ratio of it's height and width. The above lemma is a key ingredient in proving that with high probability at each step the aspect ratios of the transformed rectangles are not increasing arbitrarily.

Lemma 4.5. *There is a constant L such that on the set A_n the edge length of any non degenerate transformed dyadic square $\tilde{S} \in \mathcal{S}_l$, $1 \leq l \leq r$ is bounded below by $L^{-1}2^{-l}$ and above by $L2^{-l}$.*

Proof. Let $S \in \mathscr{S}_l$ be a dyadic square, which contains at least one sample point, and $\tilde{S} \in \mathcal{S}_l$ be the transformed S after $2l$ steps. Now for each $0 \leq i \leq l-1$, S is inside of some dyadic square $D_i \in \mathcal{S}_i$ ($D_0 = [0,1]^2$). At $2i^{th}$ step when the interior of D_i is changed due to transformation, then height and width of S is also changed accordingly. In the figure 4.2 we see that if h_i and

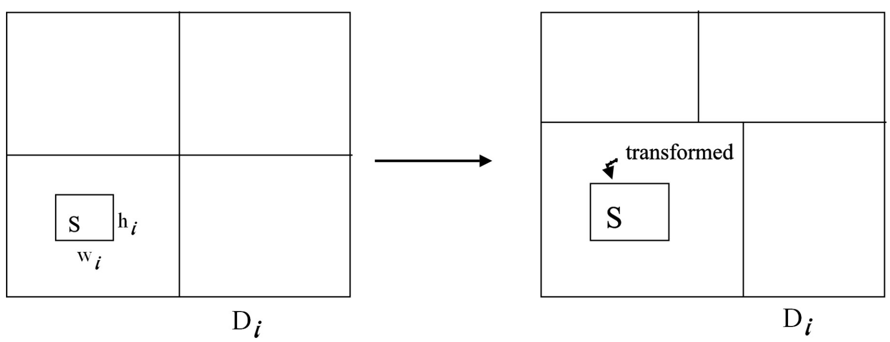


Figure 4.2: *Change of S during transformation*

w_i are respectively height and width of a dyadic square S at $2i^{th}$ step ($h_0 = w_0 = 2^{-l}$), then at the next step the height of that square is going to be one of $h_i p_{i,D_i}$ or $h_i(1-p_{i,D_i})$, and the width of that square will be one of $w_i p^*_{i,H}$, $w_i p^*_{i,H^c}$, $w_i(1-p^*_{i,H})$ or $w_i(1-p^*_{i,H^c})$, depending on which half of D_i contains that dyadic square S. Hence after $2l$ number of steps each side of S can be multiplied by at most

$$\prod_{j=1}^{l}\left(\frac{1}{2} + Lj^{\frac{1}{2}}n^{-\frac{1}{2}}2^{\frac{r+j}{2}}\right) = 2^{-l}\prod_{j=1}^{l}\left(1 + 2Lj^{\frac{1}{2}}n^{-\frac{1}{2}}2^{\frac{r+j}{2}}\right), \tag{4.8}$$

and at least

$$\prod_{j=1}^{l}\left(\frac{1}{2} - Lj^{\frac{1}{2}}n^{-\frac{1}{2}}2^{\frac{r+j}{2}}\right) = 2^{-l}\prod_{j=1}^{l}\left(1 - 2Lj^{\frac{1}{2}}n^{-\frac{1}{2}}2^{\frac{r+j}{2}}\right), \tag{4.9}$$

Note that here we are using the $L = \frac{5}{2}$ as in Lemma 4.4. Therefore we have $2Lj^{\frac{1}{2}}n^{-\frac{1}{2}}2^{\frac{r+j}{2}} \leq 2L\left(\frac{r}{n}\right)^{\frac{1}{2}}2^r \leq \frac{5}{10}\left(\frac{\log n}{n}\right)^{\frac{1}{2}}\left(\frac{n}{\log n}\right)^{\frac{1}{2}} = \frac{1}{2}$. Now we see that

$$\begin{aligned}\sum_{j=1}^{l} j^{\frac{1}{2}}n^{-\frac{1}{2}}2^{\frac{r+j}{2}} &\leq \left(\frac{r}{n}\right)^{\frac{1}{2}}\sum_{j=1}^{r}2^{\frac{j+r}{2}}\\ &\leq L\left(\frac{\log n}{n}\right)^{\frac{1}{2}}2^r\\ &= \frac{L}{6}.\end{aligned}$$

Hence using the facts that $1 + x \leq e^x \quad \forall\ x$ and $1 - x \geq e^{-2x}$ for $0 \leq x \leq \frac{1}{2}$ we get the result. □

Pick any point $u \in [0,1]^2$ and define $D_j := D_j(u)$ as the shift of u at jth step.

Lemma 4.6. *There is a constant L such that for all j of odd parity, $1 \leq j \leq 2r$, for all $\beta > 0$ and $u \in [0,1]^2$*

$$\mathbb{E}_{A_n}\left[\exp\left(\beta D_j(u)\right)|D_1,\ldots,D_{j-2}\right] \leq \exp\left(\frac{L\beta^2}{n}\right).$$

Proof. Let $u \in \tilde{S} \in \mathcal{S}_{\frac{j-1}{2}}$, where $\tilde{S}$ is one of the transformed dyadic square at $(j-1)$th step and there are 2^{j-1} such squares. Let u belongs to the upper half of $\tilde{S}$ (say H). Let L_j be the length of the vertical side of $\tilde{S}$. Take $C := C(u) \leq 1$ as the constant prescribing the position of u relative to the horizontal bisector of $\tilde{S}$. Bigger C means u is nearer to the horizontal bisector. So the displacement of u at jth step conditioned on $D_1, D_3, \ldots, D_{j-2}$ is $C(p_j - \frac{1}{2})L_j$, where $p_j := p_{j,\tilde{S}}$ is the proportion of points of $\tilde{S}$ which are inside H. Let us denote the number of points inside $\tilde{S}$ and H by M_j and X_j respectively. We observe that $M_j \sim \text{Bin}(n, 2^{-(j-1)})$, and $X_j|M_j \sim \text{Bin}(M_j, \frac{1}{2})$. Therefore conditioned on $D_1, D_3, \ldots, D_{j-2}$ and M_j the distribution of $D_j(u)$ is

$$\frac{CL_j}{M_j}\left(\text{Bin}\left(M_j, \frac{1}{2}\right) - \frac{M_j}{2}\right).$$

Now if ϵ is a bernoulli random variable satisfying $\mathbb{P}(\epsilon = 1) = \mathbb{P}(\epsilon = -1) = \frac{1}{2}$, then

$$\mathbb{E}[\exp(\lambda\epsilon)] = \frac{e^{\lambda} + e^{-\lambda}}{2} \leq e^{\frac{\lambda^2}{2}}.$$

Therefore we have

$$\begin{aligned}\mathbb{E}[\exp(\beta D_j(u))|D_1,\ldots,D_{j-2},M_j] &\leq \exp\left(\frac{C^2\beta^2L_j^2}{8M_j}\right)\\ &\leq \exp\left(\frac{\beta^2L_j^2}{8M_j}\right),\end{aligned}$$

where in the last inequality we have use the fact that $C = C(u) \le 1$. Now if L_j' denotes the length of the horizontal side of $\tilde{S}$, then $L_j L_j' = \frac{M_j}{n}$, since area of $\tilde{S}$ is proportional to the number of sample points inside it. By Lemma 4.5 we have $\frac{1}{L} \le \frac{L_j}{L_j'} \le L$ on A_n, which gives us that on A_n

$$L_j^2 \le \frac{LM_j}{n}.$$

Hence we get

$$\mathbb{E}_{A_n}[\exp(\beta D_j(u))|D_1, \ldots, D_{j-2}] \le \exp\left(\frac{L\beta^2}{n}\right).$$

□

Lemma 4.7. *We have*

$$\mathbb{E}_{A_n}\left[\exp\left(\beta \sum_{i=1,\, i\ odd}^{2r-1} D_i\right)\right] \le \exp\left(\frac{Lr\beta^2}{n}\right).$$

Proof. Using condition expectations and the previous lemma recursively, we get that

$$\begin{aligned}
\mathbb{E}_{A_n}\left[\exp\left(\beta \sum_{i=1,\, i\,\mathrm{odd}}^{2r-1} D_i\right)\right] &= \mathbb{E}\left[\mathbb{E}_{A_n}\left[\exp\left(\beta \sum_{i=1,\, i\,\mathrm{odd}}^{2r-1} D_i\right) \Bigg| D_1, D_3, \ldots, D_{2r-3}\right]\right] \\
&= \mathbb{E}_{A_n}\left[\exp\left(\beta \sum_{i=1,\, i\,\mathrm{odd}}^{2r-3} D_i\right)\right] \mathbb{E}_{A_n}\left[\exp(\beta D_{2r-1})|D_1, D_3, \ldots, D_{2r-3}\right] \\
&\le \exp\left(\frac{L\beta^2}{n}\right) \mathbb{E}_{A_n}\left[\exp\left(\beta \sum_{i=1,\, i\,\mathrm{odd}}^{2r-3} D_i\right)\right] \\
&\le \exp\left(\frac{Lr\beta^2}{n}\right).
\end{aligned}$$

□

As a consequence of this lemma, we get that for any $t > 0$

$$\mathbb{P}\left\{\left|\sum_{i=1,\, i\,\mathrm{odd}}^{2r-1} D_i\right| > t,\ A_n\right\} \le 2\exp\left(\frac{Lr\beta^2}{n} - t\beta\right).$$

Optimizing over β i.e., taking $\beta = \frac{nt}{2Lr}$ we obtain

$$\mathbb{P}\left\{\left|\sum_{i=1,\, i\,\mathrm{odd}}^{2r-1} D_i\right| > t,\ A_n\right\} \le 2\exp\left(-\frac{t^2 n}{4Lr}\right).$$

Similarly, one can do the above analysis for even i and get that for even i

$$\mathbb{P}\left\{\left|\sum_{i=2,\, i\,\mathrm{even}}^{2r} D_i\right| > t,\ A_n\right\} \le 2\exp\left(-\frac{t^2 n}{4Lr}\right).$$

Let $D(u) := \|u - \tilde{u}\|$ be the total displacement of u up to step $2r$, then we observe that

$$|D(u)|^2 = \left| \sum_{\substack{i=1 \\ i \text{ odd}}}^{2r-1} D_i(u) \right|^2 + \left| \sum_{\substack{i=2 \\ i \text{ even}}}^{2r} D_i(u) \right|^2 .$$

Hence using the above equations and fact that

$$\{|D(u)| > t\} \subset \left\{ \left| \sum_{\substack{i=1 \\ i \text{ odd}}}^{2r-1} D_i(u) \right| > \frac{t}{\sqrt{2}} \right\} \cup \left\{ \left| \sum_{\substack{i=2 \\ i \text{ even}}}^{2r} D_i(u) \right| > \frac{t}{\sqrt{2}} \right\},$$

we get that

$$\mathbb{P}(|D(u)| > t,\ A_n) \leq 4 \exp\left(-\frac{t^2 n}{8Lr} \right).$$

This gives us

$$\begin{aligned} \mathbb{E}_{A_n}\left[\exp\left(\frac{nD(u)^2}{16Lr} \right) \right] &= \int_0^\infty \mathbb{P}\left(\frac{nD(u)^2}{16Lr} > t,\ A_n \right) e^t \, dt \\ &\leq 4 \int_0^\infty e^{-t} \, dt \\ &\leq 32. \end{aligned}$$

Using Fubini's theorem we get

$$\mathbb{E}_{A_n}\left[\int\int_{[0,1]^2} \exp\left(\frac{nD(u)^2}{rL} \right) \, du \right] \leq 32, \tag{4.10}$$

(replacing $16L$ in the previous equation by L).

Recall that $\mathscr{S}_r := \{S_1, \ldots, S_{2^{2r}}\}$ is the collection of 2^{2r} dyadic squares having side length 2^{-r} (note that these are not transformed squares). Let N_i be the number of sample points in S_i, $1 \leq i \leq 2^{2r}$. Define $\hat{N} := (N_1, \ldots, N_{2^{2r}})$. We observe that for $i = 1, \ldots, 2^{2r}$,

$$\mathbb{P}(X_1 \in S_i | \hat{N}) = \frac{N_i}{n}.$$

Therefore conditional density of the random variable X_1 has the form

$$f_{X_1}(u) = \sum_{i=1}^{2^{2r}} \frac{N_i}{n} 2^{2r} \mathbf{1}_{S_i}(u).$$

But $\frac{N_i}{n}$ is the area of the transformed square $\tilde{S}_i$, and by Lemma 4.5 it is at most $L2^{-2r}$ on the set A_n. Therefore

$$f_{X_1}(u) \leq L \sum_{i=1}^{2^{2r}} \mathbf{1}_{S_i}(u) = L.$$

Then using (4.10) we see that

$$\begin{aligned}
\mathbb{E}_{A_n}\left[\exp\left(\frac{nD(X_1)^2}{rL}\right)\right] &= \mathbb{E}\left[\mathbb{E}_{A_n}\left[\exp\left(\frac{nD(X_1)^2}{rL}\right)\Bigm| \hat{N}\right]\right] \\
&= \mathbb{E}_{A_n}\left[\int\int_{[0,1]^2}\exp\left(\frac{nD(u)^2}{rL}\right)f_{X_1}(u)\,du\right] \\
&\leq L\mathbb{E}_{A_n}\left[\int\int_{[0,1]^2}\exp\left(\frac{nD(u)^2}{rL}\right)\,du\right] \\
&\leq 32L.
\end{aligned}$$

There is nothing special about X_1. The above is true for any X_i hance we get that

$$\mathbb{E}_{A_n}\left[\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{nD(X_i)^2}{rL}\right)\right] \leq L.$$

Recall that $2^{-r} = 10\left(\frac{\log n}{n}\right)^{\frac{1}{2}}$, which gives us $r \leq \log n$. Hence we get

$$\mathbb{E}_{A_n}\left[\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{n\|X_i-\tilde{X}_i\|^2}{L\log n}\right)\right] \leq L.$$

We can do the similar thing for Y_is' and get another set B_n such that $\mathbb{P}(B_n^c) \leq \frac{1}{2n^2}$, and

$$\mathbb{E}_{B_n}\left[\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{n\|Y_i-\tilde{Y}_i\|^2}{L\log n}\right)\right] \leq L.$$

□

Proof of (4.7). It is sufficient to show that on the set $A_n \cap B_n$ there exists a permutation $\pi \in \mathcal{P}(n)$ such that

$$\max_{1\leq i\leq n}\|\tilde{X}_i - \tilde{Y}_{\pi(i)}\| \leq L\left(\frac{\log n}{n}\right)^{\frac{1}{2}}.$$

Recall that $\hat{R}_i$ is the rectangle (transformed squares) in which $\hat{X}_i$ belongs to, and $\hat{B}_i$ is the similar for $\hat{Y}_i$. Now we 'match' $\hat{X}_i$ and $\hat{Y}_j$ if $|\hat{R}_i \cap \hat{B}_j| \neq 0$, where $|\cdot|$ is the lebesgue measure in $\mathbb{R}^2$. Thus we get a bipartite graph between $\{\hat{X}_i\}$ and $\{\hat{Y}_i\}$, but degree of each vertex is more than or equal to one. We want to construct a bipartite graph out of it such that degree of each vertex is one. We need Hall's marriage theorem to fulfil our purpose.

Theorem 4.8 (**Hall's marriage theorem**)**.** *Consider the set $S = \{1,\ldots,n\}$. Suppose for each $i \in S$ we can associate $A(i) \subset S$ such that for any $I \subset S$*

$$\mathit{card}\,(\cup_{i\in I}A(i)) \geq \mathit{card}(I).$$

Then there exists a permutation π of the numbers $1,\ldots,n$ such that

$$\forall\, i \in S, \quad \pi(i) \in A_i.$$

Now consider an $\hat{X}_i$, and let $A(i) = \{j : \hat{B}_j \cap \hat{R}_i \neq \phi\}$. Take any $I \subset \{1, \ldots, n\}$, and define $J := \cup_{i \in I} A(i)$. Then we claim that

$$\operatorname{card}(J) \geq \operatorname{card}(I).$$

We see that the union $\cup_{i \in I} \hat{R}_i$ only intersects the union $\cup_{j \in J} \hat{B}_j$. This implies that $\cup_{i \in I} \hat{R}_i \subset \cup_{j \in J} \hat{B}_j$. But since we know that area of each $\hat{R}_i$ and $\hat{B}_i$ is $\frac{1}{n}$, therefore we get

$$\frac{\operatorname{card}(I)}{n} = \operatorname{area}\left(\cup_{i \in I} \hat{R}_i\right) \leq \operatorname{area}\left(\cup_{j \in J} \hat{B}_j\right) = \frac{\operatorname{card}(J)}{n}.$$

Hence the claim. So, by Hall's marriage theorem, there exists a perfect matching between $\hat{X}_i$s' and $\hat{Y}_j$s' i.e., there exists a permutation $\pi \in \mathcal{P}(n)$ such that $\hat{X}_i$ is matched to $\hat{Y}_{\pi(i)}$. Using the Lemma 4.5 and the fact that $\hat{R}_i \subset \tilde{R}_i$ and $\hat{B}_j \subset \tilde{B}_j$, we can say that the diameters of $\tilde{R}_i$ and $\tilde{B}_j$ are uniformly bounded by $L\left(\frac{\log n}{n}\right)^{\frac{1}{2}}$. Hence for all $1 \leq i, j \leq n$

$$\|\tilde{X}_i - \hat{X}_i\| \leq L\left(\frac{\log n}{n}\right)^{\frac{1}{2}} \quad \text{and} \quad \|\tilde{Y}_j - \hat{Y}_j\| \leq L\left(\frac{\log n}{n}\right)^{\frac{1}{2}}.$$

On the other hand, $\|\hat{X}_i - \hat{Y}_{\pi(i)}\|$ can be at most the sum of diameters of $\hat{R}_i$ and $\hat{B}_{\pi(i)}$. But that sum is bounded by the sum of the diameters of $\tilde{R}_i$ and $\tilde{B}_{\pi(i)}$. Hence again using Lemma 4.5 we get that for any $1 \leq i \leq n$

$$\|\hat{X}_i - \hat{Y}_{\pi(i)}\| \leq L\left(\frac{\log n}{n}\right)^{\frac{1}{2}}.$$

Combining the above two equations we get our result. □

Chapter 5

The Leighton-Shor grid matching theorem

In this chapter we will discuss about another matching called min-max matching. Take n blue points and n red points uniformly and independently chosen in a unit square $[0,1]^2$. Do a matching between the blue and red points, and look at the maximal matched edge length. If we are given a realization of n sample blue points and n sample red points then we can do $n!$ many matchings out of it and have $n!$ many maximal matched edge lengths. Our interest is to investigate about the minimum of those maximal matched edge lengths. Looking at the definition of T_p in Chapter 4, we see that this is basically T_∞.

Instead of directly looking at the matching between $\{X_i\}$ and $\{Y_i\}$, we go via a middleman. Let $n \in (k^2, (k+1)^2]$ be a positive integer. We say n points $Z_1, \ldots, Z_n$ are evenly spaced in $[0,1]^2$ if the balls $B\left(Z_i, \frac{1}{2(k+1)}\right)$ of radius $\frac{1}{2(k+1)}$ around Z_i do not intersect each other and the boundary of $[0,1]^2$. If n is a perfect square, then we can easily do it by placing Z_is' at the vertices of a grid of edge length $\frac{1}{\sqrt{n}}$.

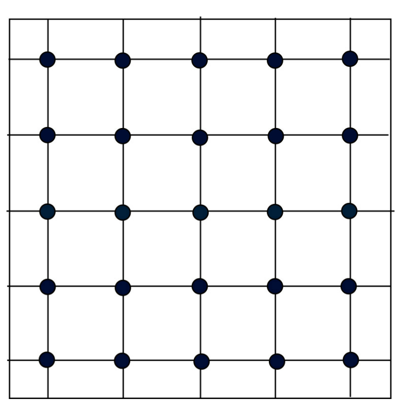


Figure 5.1: *Twenty five evenly spaced points at the vertices of a grid of mesh length $\frac{1}{5}$.*

All probabilistic theorems, lemmas, results in this chapter are asymptotic of n. So they are true for only sufficiently large n.

Theorem 5.1. *If the points $\{Z_i\}_{i\le n}$ are evenly spaced and if $\{X_i\}_{i\le n}$ are i.i.d. uniform in $[0,1]^2$, then there exists a constant $L>0$ such that with probability at least $1-L\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)$ we have*

$$\inf_{\pi\in\mathcal{P}(n)}\max_{i\le n}\|X_i-Z_{\pi(i)}\|\le L\frac{(\log n)^{\frac{3}{4}}}{\sqrt{n}}, \tag{5.1}$$

and thus

$$\mathbb{E}\left[\inf_{\pi\in\mathcal{P}(n)}\max_{i\le n}\|X_i-Z_{\pi(i)}\|\right]\le L\frac{(\log n)^{\frac{3}{4}}}{\sqrt{n}}, \tag{5.2}$$

where $\mathcal{P}(n)$ is the set of all permutations of the numbers $1,\dots,n$.

From now onwards L will denote a constant and may change value from line to line unless otherwise stated. As a consequence of this theorem we get the following

Corollary 5.2. *Let $X_1,\dots,X_n,Y_1,\dots,Y_n$ be i.i.d. uniform points in $[0,1]^2$. Then there exists a constant $L>0$ such that with probability at least $1-L\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)$ we have,*

$$\inf_{\pi\in\mathcal{P}(n)}\max_{i\le n}\|X_i-Y_{\pi(i)}\|\le L\frac{(\log n)^{\frac{3}{4}}}{\sqrt{n}}. \tag{5.3}$$

Proof. Take $\{Z_i\}_{i\le n}$ as the set of evenly spaced points in $[0,1]^2$. Let us say the matching π_1 between $\{X_i\}$, $\{Z_i\}$ minimizes the maximum edge length and the matching π_2 does the same for $\{Y_i\}$, $\{Z_i\}$. We see that these two matchings induces a matching π between $\{X_i\}$ and $\{Y_i\}$ in a natural way. We say that X_i is matched to $Y_{\pi(i)}$ if X_i and $Y_{\pi(i)}$ are matched to a common Z_k under the matchings π_1 and π_2. Using triangle inequality we get

$$\begin{aligned}\|X_i-Y_{\pi(i)}\| &\le \|X_i-Z_k\|+\|Y_{\pi(i)}-Z_k\|\\ &\le \max_{1\le i\le n}\|X_i-Z_{\pi_1(i)}\|+\max_{1\le i\le n}\|Y_i-Z_{\pi_2(i)}\|\\ &= \inf_{\pi\in\mathcal{P}(n)}\left(\max_{1\le i\le n}\|X_i-Z_{\pi(i)}\|\right)+\inf_{\pi\in\mathcal{P}(n)}\left(\max_{1\le i\le n}\|Y_i-Z_{\pi(i)}\|\right).\end{aligned}$$

Therefore we have

$$\min_{\pi\in\mathcal{P}(n)}\left(\max_{1\le i\le n}\|X_i-Y_{\pi(i)}\|\right)\le\inf_{\pi\in\mathcal{P}(n)}\left(\max_{1\le i\le n}\|X_i-Z_{\pi(i)}\|\right)+\inf_{\pi\in\mathcal{P}(n)}\left(\max_{1\le i\le n}\|Y_i-Z_{\pi(i)}\|\right).$$

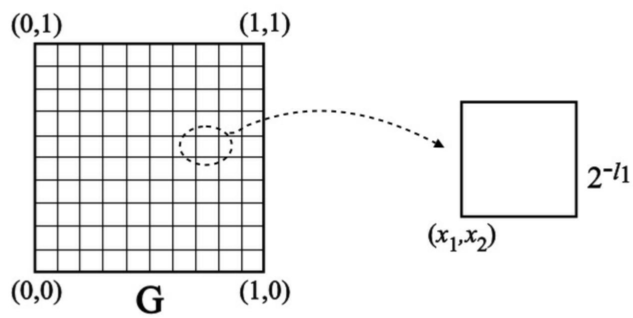
Hence we get our result. ∎

It is shown in [4] that the inequality in (5.2) can also be reversed.

In the Chapter 4 we have seen that the optimal average matched edge length is $\Theta\left(\sqrt{n \log n}\right)$. But from the previous theorem we see that it is not the case for optimal maximum matched edge length. Before proving this theorem we need some ground work. First we divide the unit square into a grid. To do so, let l_1 be the largest integer such that $2^{-l_1} \geq \frac{(\log n)^{\frac{3}{4}}}{\sqrt{n}}$. Now define the grid G of *mesh length* 2^{-l_1} as,

$$G = \{(x_1, x_2) \in [0,1]^2 : 2^{l_1}x_1 \in \mathbb{N} \textit{ or } 2^{l_1}x_2 \in \mathbb{N}\}.$$

We observe that $(x_1, x_2) \in [0,1]^2$ is a *vertex* of the grid G if $2^{l_1}x_1 \in \mathbb{N}$ and $2^{l_2}x_2 \in \mathbb{N}$, and



according to the construction an *edge* of the grid G is the line segment between two vertices which are at a distance 2^{-l_1} apart from each other. A *grid square* is a square made of edges of length 2^{-l_1}.

In our proof we need to consider regions made of grid squares. Define a *simple curve* as a image of a continuous map $\phi : [0,1] \to \mathbb{R}^2$ such that ϕ is one-to-one on $[0,1)$. Quite naturally we say that a simple curve is *traced on* G if $\phi([0,1]) \subset G$ and it is *closed* if $\phi(0) = \phi(1)$. We see that a simple closed curve C divides the $\mathbb{R}^2$ into two parts. One is the bounded one and another one is the unbounded. Call the interior of the bounded region as $\mathring{C}$ and interior of the unbounded region intersected with $[0,1]^2$ as $\mathring{C}^c$. Now we will move to the proof of Theorem 5.1 and the key ingredient of the proof is the following proposition.

Proposition 5.3. *With probability at least* $1 - L\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)$ *the following occurs . Given any simple closed curve C traced on G, we have*

$$\left|\sum_{i=1}^{n}\left(\mathbf{1}_{\mathring{C}}(X_i) - \lambda(\mathring{C})\right)\right| \leq Ll(C)\sqrt{n}(\log n)^{\frac{3}{4}},$$

where $l(C)$ is the length of C and $\lambda(\mathring{C})$ is the area of $\mathring{C}$.

Before moving into the proof of Proposition 5.3 let us see how does it imply the Theorem 5.1.

Proof of Theorem 5.1 from Proposition 5.3. First we define a notion called *chord*. We say that a simple curve C traced on G is a chord of $[0,1]^2$ if it is image of a continuous map $\phi : [0,1] \to G$ such that $\phi(0)$ and $\phi(1)$ belong to boundary of $[0,1]^2$. We see that a chord C divides $[0,1]^2$ into two parts, say R_1 and R_2 and we have,

$$\begin{aligned}
\sum_{i=1}^{n} \left(\mathbf{1}_{[0,1]^2}(X_i) - \lambda([0,1]^2)\right) &= 0 \\
\Rightarrow \sum_{i=1}^{n} (\mathbf{1}_{R_1}(X_i) - \lambda(R_1)) + \sum_{i=1}^{n} (\mathbf{1}_{R_2}(X_i) - \lambda(R_2)) &= 0 \\
\Rightarrow \sum_{i=1}^{n} (\mathbf{1}_{R_1}(X_i) - \lambda(R_1)) &= -\sum_{i=1}^{n} (\mathbf{1}_{R_2}(X_i) - \lambda(R_2)).
\end{aligned}$$

We define,

$$\mathcal{D}(C) = \left| \sum_{i=1}^{n} (\mathbf{1}_{R_1}(X_i) - \lambda(R_1)) \right|.$$

Now we observe that given a chord C, there exists a simple closed curve C' traced on G such that $\mathring{C}' = R_1$ or $\mathring{C}' = R_2$ and $l(C') \leq 4l(C)$. Therefore as a consequence of Proposition 5.3 we have the following result.

Result 5.1. *With probability at least* $1 - L\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)$, *for each chord C we have*

$$\mathcal{D}(C) \leq 4Ll(C)\sqrt{n}(\log n)^{\frac{3}{4}}.$$

Now we construct a coarser grid $G' \subset G$ of mesh length 2^{-l_2}, where $l_2 \leq l_1$, to be determined later. Given a region R formed by union of squares of G', we construct a new region R' as the union of squares of G' such that at least one edge of the squares contained in R. Our main goal is to prove the following result

$$n\lambda(R') \geq \mathrm{card}\{i \leq n : X_i \in R\}. \tag{5.4}$$

We say that a domain R is decomposable if it can be written as $R = R_1 \cup R_2$, where both R_1 and R_2 are union of squares of G' and $R_1 \cap R_2$ is a finite set or equivalently R_1 and R_2 intersects at their vertices only. Given any region R in G' we can write it as a union of undecomposable regions i.e., $R = \cup_{i=1}^{k} R_i$, where $R_1, \ldots, R_k$ are not decomposable (see figure 5.2). Now, we want to figure out the relationship between R' and R_i's. At least one thing we can observe that

$$\lambda(R' \backslash R) \leq \sum_{i=1}^{k} \lambda(R_i' \backslash R_i).$$

We observe that any square in $R' \backslash R$ can be counted at most four times during counting of squares of $R_i' \backslash R_i$ (see figure 5.2). Therefore we have

$$\frac{1}{4} \sum_{i=1}^{k} \lambda(R_i' \backslash R_i) \leq \lambda(R' \backslash R). \tag{5.5}$$

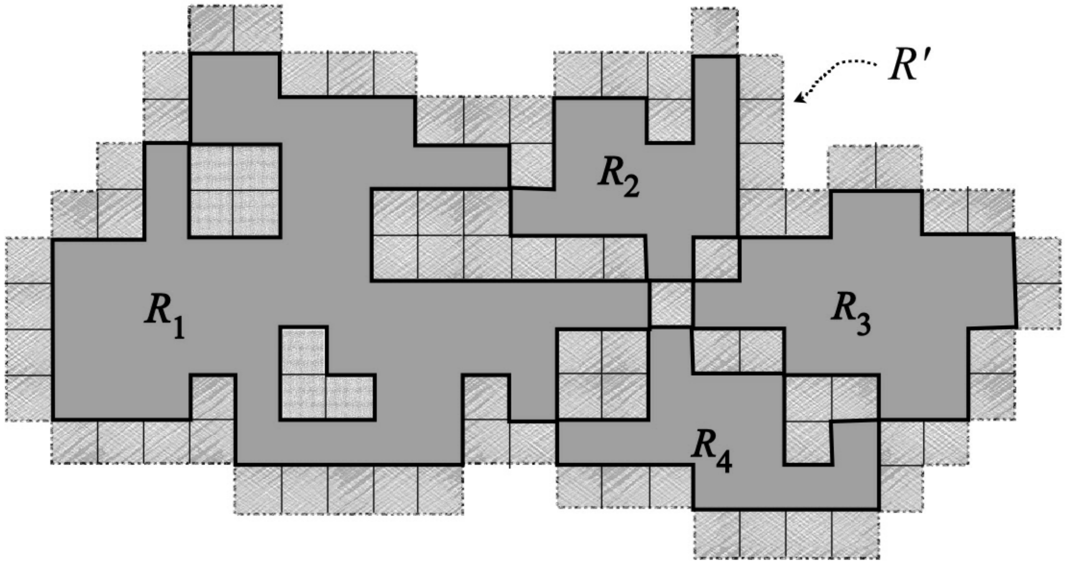


Figure 5.2: *Decomposable region R and corresponding R'*

Hence to prove (5.4) it suffices to prove that for any undecomposable region R

$$\mathrm{card}\{i \leq n : X_i \in R\} - n\lambda(R) \leq \frac{1}{4}\lambda(R'\backslash R). \tag{5.6}$$

Let S be the boundary of an undecomposable region R. We observe that if a vertex $w \in S$ then either two or four edges of G' adjacent to w belong to S. Moreover the boundary S can be decomposed as a union of simple closed curves $C_1, \ldots, C_k$ (see figure 5.3). Next we see that for any $l \leq k$ either $R \subset \mathring{C}_l$ or $R \cap \mathring{C}_l = \phi$. We observe that there is only one C_l for which $R \subset \mathring{C}_l$. Without loss of generality take that C_l as C_1 and call it as *outer boundary.* For other C_ls, $R \cap \mathring{C}_l = \phi$, which implies that those C_ls come from holes inside R_1 and finally we get that

$$R = \mathring{C}_1 \backslash \cup_{l=2}^{k} \mathring{C}_l. \tag{5.7}$$

Let $\tilde{R}_l$ be the set of squares of which at least one edge is contained in $\mathring{C}_l$. Then using the similar kind of argument as in (5.5) we obtain

$$\frac{1}{4}\sum_{l=1}^{k} \lambda(\tilde{R}_l \backslash R) \leq \lambda(R'\backslash R). \tag{5.8}$$

We see that $\mathrm{card}\{i \leq n : X_i \in R\} = \mathrm{card}\{i \leq n : X_i \in \mathring{C}_1\} - \sum_{l=2}^{k} \mathrm{card}\{i \leq n : X_i \in \mathring{C}_l\}$ and $\lambda(R) = \lambda(\mathring{C}_1) - \sum_{l=2}^{k} \lambda(\mathring{C}_l)$. Thus we get that

$$|\mathrm{card}\{i \leq n : X_i \in R\} - n\lambda(R)| \leq \sum_{l=1}^{k} \left|\mathrm{card}\{i \leq n : X_i \in \mathring{C}_l\} - n\lambda(\mathring{C}_l)\right|.$$

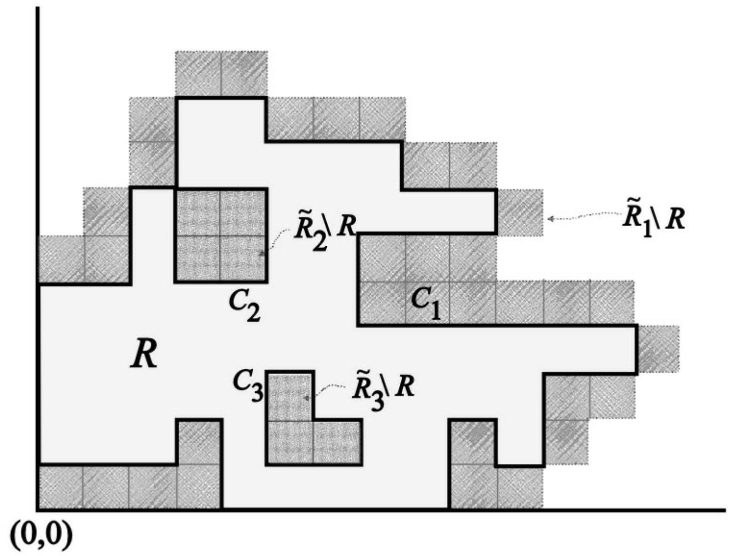


Figure 5.3: *Undecomposable region R and its boundary*

Therefore to prove (5.6), it suffices to show that for each $1 \leq l \leq k$ we have

$$\left|\operatorname{card}\{i \leq n : X_i \in \mathring{C}_l\} - n\lambda(\mathring{C}_l)\right| \leq n2^{-4}\lambda(\tilde{R}_l\backslash R). \tag{5.9}$$

This can be easily proved from Proposition 5.3. For $l \geq 2$, C_l does not intersect the boundary of $[0,1]^2$. Therefore $\tilde{R}_l\backslash R$ contains at least $\frac{1}{4}\frac{l(C_l)}{2^{-l_2}}$ squares. Because for a simplest looking C_l (see C_2 in figure 5.3) $\tilde{R}_l\backslash R$ contains exactly $\frac{l(C_l)}{2^{-l_2}}$ squares. Basically we are counting the square in $\tilde{R}_l\backslash R$ by the number of grid edges of G' required to form C_l (that is exactly $\frac{l(C_l)}{2^{-l_2}}$), but to do so we may count one square in $\tilde{R}_l\backslash R$ at most three times (see C_3 in figure 5.3). Therefore we have

$$\lambda(\tilde{R}_l\backslash R) \geq \frac{1}{3}\frac{l(C_l)}{2^{-l_2}}2^{-2l_2} \geq \frac{1}{4}\frac{l(C_l)}{2^{l_2}}. \tag{5.10}$$

If we take l_2 in such a way that

$$\frac{2^9 L}{\sqrt{n}}(\log n)^{\frac{3}{4}} \geq 2^{-l_2} \geq \frac{2^8 L}{\sqrt{n}}(\log n)^{\frac{3}{4}}, \tag{5.11}$$

then from Proposition 5.3, (5.11), and (5.10) it follows that,

$$\begin{aligned}\left|\operatorname{card}\{i \leq n : X_i \in \mathring{C}_l\} - n\lambda\mathring{C}_l\right| &\leq Ll(C_l)\sqrt{n}(\log n)^{\frac{3}{4}} \\ &\leq \frac{nl(C_l)}{2^{8+l_2}} \\ &\leq n2^{-4}\lambda(\tilde{R}_l\backslash R).\end{aligned}$$

But for $l = 1$, C_1 is the outer boundary and (5.10) need not be true. Because C_1 might trace the boundary of $[0,1]^2$, for example if C_1 is the complete boundary of $[0,1]^2$ then $\lambda(\tilde{R}_1 \backslash R) = \phi$ but $\frac{1}{4}\frac{l(C_1)}{2^{l_2}} = 2^{-l_2-2} \neq 0$. In that case we will use the Result 5.1 to get (5.9). We see that C_1 is the union of parts of boundary of $[0,1]^2$ and chords of $[0,1]^2$. Let us say that $C_1 = \cup_{i=1}^{k_1} B_i \cup_{j=1}^{k_2} \tilde{C}_j^h$, where B_is' are the parts of the boundary of $[0,1]^2$ and $\tilde{C}_j^h$s' are the chords of $[0,1]^2$. Let us define the region R_j^h as the component of $\mathring{C}_1^c$ which shares the chord $\tilde{C}_j^h$ (see figure 5.4).

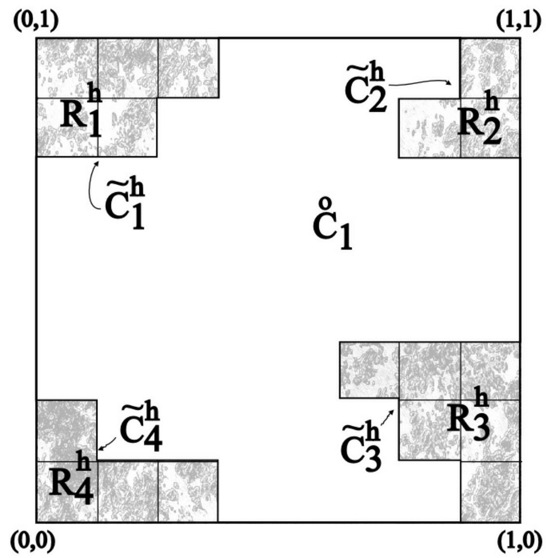


Figure 5.4: C_1 *and* $R_j^h s$

We observe that,

$$\left[\text{card}\{i \leq n : X_i \in \mathring{C}_1\} - n\lambda(\mathring{C}_1)\right] + \sum_{j=1}^{k_2} \left[\text{card}\{i \leq n : X_i \in R_j^h\} - n\lambda(R_j^h)\right] \quad = \quad 0$$

$$\Rightarrow \left|\text{card}\{i \leq n : X_i \in \mathring{C}_1\} - n\lambda(\mathring{C}_1)\right| \leq \sum_{j=1}^{k_2} \left|\text{card}\{i \leq n : X_i \in R_j^h\} - n\lambda(R_j^h)\right|.$$

Now by the Result 5.1 we have,

$$\left|\text{card}\{i \leq n : X_i \in R_j^h\} - n\lambda(R_j^h)\right| = \mathcal{D}(C_j^h) \leq 4Ll(C_j^h)\sqrt{n}(\log n)^{\frac{3}{4}}.$$

Hence we have,

$$\begin{aligned}\left|\text{card}\{i \leq n : X_i \in \mathring{C}_1\} - n\lambda(\mathring{C}_1)\right| &\leq 4L\sqrt{n}(\log n)^{\frac{3}{4}} \sum_{j=1}^{k_2} l(C_j^h) \\ &\leq \frac{n2^{-l_2}}{2^6} \sum_{j=1}^{k_2} l(C_j^h).\end{aligned}$$

Using the same argument as used for C_ls' for $l \geq 2$, we see that

$$\lambda(\tilde{R}_1 \backslash R) \geq \frac{1}{4} 2^{-l_2} \sum_{j=1}^{k_2} l(C_j^h),$$

and hence using the above two equations we obtain,

$$\left|\text{card}\{i \leq n : X_i \in \mathring{C}_1\} - n\lambda(\mathring{C}_1)\right| \leq n2^{-4}\lambda(\tilde{R}_1 \backslash R).$$

Since the points $\{Z_i\}_{i \leq n}$ are evenly spaced, therefore we can say that the points are nearly $\frac{1}{\sqrt{n}}$ distance apart from each other. Therefore if we take

$$2^{-l_2} \geq \frac{3}{\sqrt{n}} \tag{5.12}$$

then for any region R made of grid squares of G' we have,

$$\text{card}\{i \leq n : Z_i \in (R')'\} \geq n\lambda(R').$$

Hence using (5.4) we get

$$\text{card}\{i \leq n : Z_i \in (R')'\} \geq \text{card}\{i \leq n : X_i \in R\}. \tag{5.13}$$

Define,

$$A(i) = \{j \leq n : d(X_i, Z_j) \leq 4\sqrt{2}2^{-l_2}\}.$$

Take any subset I of $1, \ldots, n$ and define the region R as the union of the grid squares of G' which contains at least one X_i, $i \in I$, then from (5.13) we get,

$$\text{card}\left(\cup_{i \in I} A(i)\right) \geq \text{card}(I).$$

Now taking l_2 as the largest integer which satisfies both (5.11) and (5.12) and using Hall's marriage theorem, stated in Chapter 4, we can say that there exists a permutation $\pi \in \mathcal{P}(n)$ such that

$$\sup_{i \leq n} d(X_i, Z_{\pi(i)}) \leq 4\sqrt{2}2^{-l_2} \leq \frac{L'}{\sqrt{n}}(\log n)^{\frac{3}{4}},$$

which is our desired result. □

Now we move to the proof of Proposition 5.3. This can be derived from a simpler version of it.

Lemma 5.4. *Consider a vertex $w \in G$ and define $\mathcal{C}(w,k)$ as the set of all closed curves of length at most 2^k traced on G and passes through the vertex w. Then if $k \leq l_1 + 2$, with probability at least $1 - L\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)$, for each $C \in \mathcal{C}(w,k)$ we have,*

$$\left|\sum_{i=1}^{n} \left(\mathbf{1}_{\mathring{C}}(X_i) - \lambda(\mathring{C})\right)\right| \leq L2^k\sqrt{n}(\log n)^{\frac{3}{4}}. \tag{5.14}$$

Before proving the Lemma 5.4, we will see the derivation of Proposition 5.3 from this lemma.

Proof of Proposition 5.3 from Lemma 5.4. Since there are at most $(2^{l_1}+1)^2$ choice of vertices w and at most $(2l_1+4)$ choices of $k \in [-l_1, l_1+2] \cap \mathbb{Z}$, we can say that with probability at least

$$1 - L(2^{l_1}+1)^2(2l_1+4)\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)$$

(5.14) occurs for all choices of $C \in \mathcal{C}(w,k)$ and any $k \in [-l_1, l_1+2] \cap \mathbb{Z}$. Consider a simple closed curve C traced on G. We can bound the length of C by the total number of edges of G. Thus we have,

$$2^{-l_1} < l(C) \leq 2(2^{l_1}+1) \leq 2^{l_1+2}.$$

If we take k be the smallest integer such that $l(C) \leq 2^k$, then $2^k \leq 2l(C)$ and the Lemma 5.4 implies that with probability at least $1 - L(2^{l_1}+1)^2(2l_1+4)\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)$ for any curve C traced on G we have

$$\begin{aligned}\left|\sum_{i=1}^{n}\left(\mathbf{1}_{\mathring{C}}(X_i) - \lambda(\mathring{C})\right)\right| &\leq L2^k\sqrt{n}(\log n)^{\frac{3}{4}} \\ &\leq 2Ll(C)\sqrt{n}(\log n)^{\frac{3}{4}}.\end{aligned}$$

We observe that for a different constant $L' = 2L$,

$$\begin{aligned}\frac{L(2^{l_1}+1)^2(2l_1+4)\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)}{L'\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L'}\right)} &\leq 2^{2l_1+2}(l_1+2)\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{2L}\right) \\ &\leq 2^{2\frac{n(\log n+1)}{(\log n)^{\frac{3}{2}}}}\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{2L}\right) \\ &\to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Therefore,

$$1 - L(2^{l_1}+1)^2(2l_1+4)\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right) \geq 1 - 2L\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{2L}\right)$$

for sufficiently large n. Hence the proof. □

It remains to prove Lemma 5.4. To prove it we will use a different technique called Generic chaining introduced by Talagrand [7]. We will discuss about this in the next chapter.

Chapter 6

The generic chaining

In this chapter we will prove the Lemma 5.4 using a different technique called Generic chaining. We consider a stochastic process $\{X_t\}_{t\in T}$ indexed by a metric space (T,d). First we will analyze various properties of it. From now on in this chapter we will take $N_n := 2^{2^n}$; $n \in \mathbb{N}$, and L as a universal constant which may vary from contexts to contexts. Also $K(\alpha,\beta,\dots)$ indicates a constant, which depends on the parameters $\alpha,\beta,\dots$ mentioned inside the bracket.

Definition 6.1. Given a set T, an admissible sequence is an increasing sequence $\{\mathcal{A}_n\}$ of partitions of T such that $\text{card}(\mathcal{A}_n) \leq N_n$.

If we are provided an admissible sequence then for each fixed $t \in T$ we can define a sequence of subsets $\{A_n(t)\}$ of T such that $A_n(t) \in \mathcal{A}_n$ and $A_n(t)$ contains t.

Definition 6.2. Given $\alpha, \beta > 0$, and a metric space (T,d), we define

$$\gamma_\alpha(T,d) = \inf \sup_t \sum_{n\geq 0} 2^{\frac{n}{\alpha}} \Delta(A_n(t)),$$

and

$$\gamma_{\alpha,\beta}(T,d) = \left(\inf \sup_t \sum_{n\geq 0} \left(2^{\frac{n}{\alpha}} \Delta(A_n(t)) \right)^\beta \right)^{\frac{1}{\beta}},$$

where infimum is taken over all admissible sequences and $\Delta(A_n(t))$ is the diameter of $A_n(t)$.

In particular we observe that,

$$\gamma_{\alpha,1}(T,d) = \gamma_\alpha(T,d).$$

Definition 6.3. Given a metric space (T,d), define

$$e_n(T,d) = \inf \sup_t d(t,T_n),$$

where infimum is taken over all possible subsets T_n of T such that $\text{card}(T_n) \leq N_n$.

If we have only one metric d in our consideration then often we write $e_n(T)$ instead of writing $e_n(T,d)$.

Theorem 6.4. *Consider a set T (finite) provided with two distances d_1 and d_2. Consider a process $\{X_t\}_{t\in T}$ that satisfies $\mathbb{E}[X_t]=0$, and $\forall s,t\in T$, $\forall\, u>0$*

$$\mathbb{P}\left(|X_s-X_t|\geq u\right)\leq 2\exp\left(-\min\left(\frac{u^2}{d_2(s,t)^2},\frac{u}{d_1(s,t)}\right)\right).$$

Then for all values of $u_1,u_2>0$ and a prefixed $t_0\in T$ we have

$$\mathbb{P}\left(\sup_{t\in T}|X_t-X_{t_0}|\geq L(\gamma_1(T,d_1)+\gamma_2(T,d_2))+u_1D_1+u_2D_2\right)\quad\leq\quad L\exp\left(-\min(u_2^2,u_1)\right),$$

where $D_j:=4\sum_{n\geq 0}e_n(T,d_j)$; $j=1,2$.

Proof. From the definition of $e_n(T,d)$ we observe that there exists a finite set T_n of cardinality at most N_n such that $\sup_{t\in T}d(t,T_n)\leq 2e_n(T,d)$. From this finite set we can get a partition of T. For each $u\in T_n$ define

$$V(u):=\{t\in T: d(t,T_n)=d(t,u)\}.$$

These are called voronoi cells. Taking the collection of $V(u)$s' and modifying boundaries of each cell suitably we can get a partition of T into at most N_n sets having diameter at most $4e_n(T,d)$. Thus we get an admissible sequence $\{\mathcal{B}'_n\}$ such that

$$\forall\, B\in\mathcal{B}'_n,\quad \Delta_1(B)\leq 4e_n(T,d_1).$$

Similarly, we also get another admissible sequence $\{\mathcal{C}'_n\}$ such that

$$\forall\, C\in\mathcal{C}'_n,\quad \Delta_2(C)\leq 4e_n(T,d_2)$$

(Δ_j corresponds to the metric d_j). Now from the definition of $\gamma_\alpha(T,d_1)$ we see that there exists admissible sequences $\{\mathcal{B}_n\}$ and $\{\mathcal{C}_n\}$ such that

$$\forall\, t\in T,\quad \sum_{n\geq 0}2^n\Delta_1(B_n(t))\leq 2\gamma_1(T,d_1)$$

and

$$\forall\, t\in T,\quad \sum_{n\geq n}2^{\frac{n}{2}}\Delta_2(C_n(t))\leq 2\gamma_2(T,d_2).$$

We define a new sequence of partitions $\{\mathcal{A}_n\}$ as $\mathcal{A}_0=\mathcal{A}_1=\{T\}$ and for $n\geq 2$, $\mathcal{A}_n$ is the partition generated by $\mathcal{B}_{n-2}$, $\mathcal{B}'_{n-2}$, $\mathcal{C}_{n-2}$ and $\mathcal{C}'_{n-2}$, which means that a partition consists of sets like $B\cap B'\cap C\cap C'$ where $B\in\mathcal{B}_{n-2}$, $B'\in\mathcal{B}'_{n-2}$, $C\in\mathcal{C}_{n-2}$ and $C'\in\mathcal{C}'_{n-2}$. We see that

$$\begin{aligned}\mathrm{card}(\mathcal{A}_n) &\leq N_{n-2}^4\\ &= 2^{4.2^{n-2}}\\ &= 2^{2^n}.\end{aligned}$$

Therefore, $\{\mathcal{A}_n\}$ is an admissible sequence. Recall that $\{A_n(t)\}$ is a sequence of sets such that $t \in A_n(t) \in \mathcal{A}_n$. For each $n \geq 3$ pick a point $\pi_n(t) \in A_n(t)$, and for $n = 0, 1, 2$ take $\pi_n(t) = t_0$ (note that $\mathcal{A}_n = \{T\}$ for $n = 0, 1, 2$. So we can take $\pi_n(t) = t_0 \in T$ for $n = 0, 1, 2$). Since $\Delta_1(A_n(t)),\ \Delta_2(A_n(t)) \to 0$ as $n \to \infty$, we can assume that for $j = 1, 2$; $d_j(t, \pi_n(t)) \to 0$ as $n \to \infty$. Therefore,

$$|X_t - X_{t_0}| \leq \sum_{n \geq 0} \left| X_{\pi_n(t)} - X_{\pi_{n-1}(t)} \right|.$$

We want to get a bound for $|X_{\pi_n}(t) - X_{\pi_{n-1}(t)}|$. Take

$$U = (2^n + u_1) d_1(\pi_n(t), \pi_{n-1}(t)) + (2^{\frac{n}{2}} + u_2) d_2(\pi_n(t), \pi_{n-1}(t)).$$

We see that

$$\frac{U^2}{d_2^2(\pi_n(t), \pi_{n-1}(t))} \geq 2^n + u_2^2$$

and

$$\frac{U}{d_1(\pi_n(t), \pi_{n-1}(t))} \geq 2^n + u_1.$$

Hence from our hypothesis we get,

$$\mathbb{P}\left(\left| X_{\pi_n(t)} - X_{\pi_{n-1}(t)} \right| \geq U \right) \leq 2 \exp\left(-2^n - \min(u_2^2, u_1) \right).$$

We observe that,

$$\begin{aligned}
U &= (2^n + u_1) d_1(\pi_n(t), \pi_{n-1}(t)) + (2^{\frac{n}{2}} + u_2) d_2(\pi_n(t), \pi_{n-1}(t)) \\
&\leq (2^n + u_1) \Delta_1(A_{n-1}(t)) + (2^{\frac{n}{2}} + u_2) \Delta_2(A_{n-1}(t)) \\
&\leq 2^n \Delta_1(B_{n-3}(t)) + 2^{\frac{n}{2}} \Delta_2(C_{n-3}(t)) + 4u_1 e_{n-3}(T, d_1) + 4u_2 e_{n-3}(T, d_2) \\
&:= V.
\end{aligned}$$

We see that for each $t \in T$ and $n \geq 3$, with probability at least $1 - \exp(-2^n - \min\{u_2^2, u_1\})$; $|X_{\pi_n(t)} - X_{\pi_{n-1}(t)}| \leq V$. But we also observe that if $|X_{\pi_n(t)} - X_{\pi_{n-1}(t)}| \leq V$ for some fixed $t \in T$, then $|X_{\pi_n(t)} - X_{\pi_{n-1}(t)}| \leq V$ for all t in the same partitioning set. Since at the nth step there are at most 2^{2^n} many partitioning set, therefore with probability at least $1 - 2^{2^n} \exp(-2^n - \min\{u_2^2, u_1\}) = 1 - \exp(-c2^n) \exp(-\min\{u_2^2, u_1\})$ (where $c = 1 - \log 2 > 0$), $|X_{\pi_n(t)} - X_{\pi_{n-1}(t)}| \leq V$ for all $t \in T$. So using the fact $\pi_n(t) = t_0$ for $n = 0, 1, 2$, and taking sum over $n \geq 3$ we get that with probability at least $1 - L\exp(-\min(u_2^2, u_1))$ for all $t \in T$

$$\begin{aligned}
|X_t - X_{t_0}| &\leq \sum_{n \geq 3} \left[2^n \Delta_1(B_{n-3}(t)) + 2^{\frac{n}{2}} \Delta_2(C_{n-3}(t)) + 4u_1 e_{n-3}(T, d_1) + 4e_{n-3}(T, d_2) \right] \\
&\leq 16\left(\gamma_1(T, d_1) + \gamma_2(T, d_2) \right) + u_1 D_1 + u_2 D_2.
\end{aligned}$$

Taking the supremum over $t \in T$ we get the result. $\square$

6.1 Partitioning Scheme

Till now we are talking about partitions, admissible sequences etc. But given an abstract metric space and a stochastic process on it, we do not know how to get an admissible sequence. In this section we will discuss about this issue. Given a metric space (T, d), we consider a decreasing sequence of functionals $\{F_n\}$ on (T, d) such that

$$F_n(A) \leq F_{n-1}(A) \quad \forall\, A \subset T,$$

and

$$F_n(A) \leq F_n(A') \quad \text{if } A \subset A'.$$

Once we have the functionals, the main theorem in this section will guarantee that we can get an admissible sequence out of it. To make the things precise we need some definitions etc.

Definition 6.5. Consider a function $\theta : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$. We say that the sequence $\{\theta(n)\}$ satisfies the regularity condition if there exists $1 < \zeta \leq 2$, $r \geq 4$ and $\beta > 0$ such that

$$\zeta\theta(n) \leq \theta(n+1) \leq \frac{r^\beta}{2}\theta(n) \quad \forall\, n \geq 0. \tag{6.1}$$

For example $\theta(n) = 2^{\frac{n}{2}}$. As we said that from the functionals one can get an admissible sequence. But one needs to impose some conditions on the functionals. We define the growth condition as follows.

Definition 6.6 (Growth condition)**.** We say that the functionals F_n satisfy growth condition if for a certain integer $\tau \geq 1$, and for certain numbers $r \geq 4$, $\beta > 0$, the following holds true. Consider any integer $n \geq 0$ and set $m = N_{n+\tau}$. Then for any $s \in T$, any $a > 0$, any $t_1, \ldots, t_m$, any collection of sets H_l satisfying

(i) $t_l \in B(s, ar) \quad \forall\, 1 \leq l \leq m$

(ii) $d(t_l, t_{l'}) \geq a \quad \forall\, 1 \leq l \neq l' \leq m$

(iii) $H_l \subset B\left(t_l, \frac{a}{r}\right) \quad \forall\, 1 \leq l \leq m,$

we have

$$F_n\left(\cup_{l=1}^m H_l\right) \geq a^\beta\theta(n+1) + \min_{l \leq m} F_{n+1}(H_l). \tag{6.2}$$

The following theorem gives a way to get an admissible sequence out of functionals.

Theorem 6.7. *If we have a sequence of functionals $\{F_n\}$ satisfying the growth condition, then we can find an increasing sequence $\{\mathcal{A}_n\}$ of partitions with card$(\mathcal{A}_n) \leq N_{n+\tau}$ such that*

$$\sup_{t\in T}\sum_{n\geq 0}\theta(n)\Delta^\beta(A_n(t)) \leq L(2r)^\beta\left(\frac{F_0(T)}{\zeta - 1} + \theta(0)\Delta^\beta(T)\right),$$

where $\theta(n)$ satisfies the regularity condition described in Definition 6.5.

See that the sequence of partitions obtained using this theorem is not an admissible sequence. Because here the cardinality of $\mathcal{A}_n$ may go up to $N_{n+\tau} > N_n$. To get an admissible sequence we need the following lemma.

Lemma 6.8. *Consider $\alpha, \beta > 0$, a positive integer τ and an increasing sequence of partitions $\mathcal{B}_n$ with card$(\mathcal{B}_n) \leq N_{n+\tau}$. Let*

$$S = \sup_{t \in T} \sum_{n \geq 0} 2^{\frac{n}{\alpha}} \Delta^{\beta}(B_n(t)).$$

Then we can find an admissible sequence $\{\mathcal{A}_n\}$ such that

$$\sup_{t \in T} \sum_{n \geq 0} 2^{\frac{n}{\alpha}} \Delta^{\beta}(A_n(t)) \leq 2^{\frac{\tau}{\alpha}} \left(S + K(\alpha)\Delta^{\beta}(T) \right).$$

Proof. Set $\mathcal{A}_n = T$ for $n \leq \tau$ and for $n \geq \tau \mathcal{A}_n = \mathcal{B}_{n-\tau}$, so that card$(\mathcal{A}_n)$=card$(\mathcal{B}_{n-\tau}) \leq N_n$, and

$$\begin{aligned} \sum_{n \geq \tau} 2^{\frac{n}{\alpha}} \Delta^{\beta}(A_n(t)) &= 2^{\frac{\tau}{\alpha}} \sum_{n \geq 0} 2^{\frac{n}{\alpha}} \Delta^{\beta}\left(A_{n+\tau}(t)\right) \\ &= 2^{\frac{\tau}{\alpha}} \sum_{n \geq 0} 2^{\frac{n}{\alpha}} \Delta^{\beta}\left(B_n(t)\right). \end{aligned}$$

On the other hand using $\Delta(A_n(t)) = \Delta(T)$ for $n \leq \tau$ we get

$$\begin{aligned} \sum_{n \leq \tau} 2^{\frac{n}{\alpha}} \Delta^{\beta}(A_n(t)) &= \Delta^{\beta}(T) \sum_{n \leq \tau} 2^{\frac{n}{\alpha}} \\ &= 2^{\frac{\tau}{\alpha}} \Delta^{\beta}(T) \sum_{n \leq \tau} 2^{-\frac{n}{\alpha}} \\ &\leq K(\alpha) 2^{\frac{\tau}{\alpha}} \Delta^{\beta}(T), \end{aligned}$$

where $K(\alpha) = \sum_{n=0}^{\infty} 2^{-\frac{n}{\alpha}}$ depends only on α. Now using the above two estimates we get our desired result. $\square$

Proof of Theorem 6.7. In our proof we will consider balls of radius of type r^{-j}, $j \in \mathbb{N} \cup \{0\}$. Taking $m = N_{n+\tau}$ and $a = r^{-j-1}$, we rewrite the growth condition in the following way

Let $t_1, \ldots, t_m$ be a collection of m points in T, and $H_1, \ldots, H_m$ be a collection of subsets of T such that

(i) $t_l \in B(s, r^{-j}) \quad \forall\, 1 \leq l \leq m$

(ii) $d(t_l, t_{l'}) \geq r^{-j-1} \quad \forall\, 1 \leq l \neq l' \leq m$

(iii) $H_l \subset B\left(t_l, r^{-j-2}\right) \quad \forall\, 1 \leq l \leq m.$

Then we have

$$F_n\left(\cup_{l=1}^m H_l\right) \geq r^{-\beta(j+1)}\theta(n+1) + \min_{l\leq m} F_{n+1}(H_l).$$

Now we are going to construct our promised increasing sequence of partitions $\{\mathcal{A}_n\}$ by induction. For each $C \in \mathcal{A}_n$ we will associate a point $t_C \in T$, an integer $j(C)$ and three numbers $b_i(C)$; $i = 0, 1, 2$, such that

$$C \subset B\left(t_C, r^{-j(C)}\right). \tag{6.3}$$

In particular $\Delta(C) \leq 2r^{-j(C)}$. Also

$$F_n(C) \leq b_0(C) \tag{6.4}$$

$$\forall\, t \in C \quad F_n\left(C \cap B\left(t, r^{-j(C)-1}\right)\right) \leq b_1(C) \tag{6.5}$$

$$\forall\, t \in c \quad F_n\left(C \cap B\left(t, r^{-j(C)-2}\right)\right) \leq b_2(C). \tag{6.6}$$

We also assume that

$$b_1(C) \leq b_0(C), \tag{6.7}$$

and

$$b_0(C) - r^{-\beta(j(C)+1)}\theta(n) \leq b_2(C) \leq b_0(C) + \epsilon_n, \tag{6.8}$$

where $\epsilon_n = 2^{-n}F_0(T)$. The first assumption is not really a restriction. Because from (6.4) and (6.5), we see that one can easily take b_0 and b_1 in such a way that (6.7) will be satisfied. In the construction procedure we will take b_is' in such a way that (6.8) will be satisfied.

We start the construction with

$$\mathcal{A}_0 = \{T\},\ b_0(T) = b_1(T) = b_2(T) = F_0(T),$$

and t_T as an arbitrary point in T. We take $j(T)$ as the largest integer satisfying $T \subset B\left(t_T, r^{-j(T)}\right)$.

As we stated before that we will proceed by induction. So we assume that for a certain $n \geq 0$ we have already constructed $\mathcal{A}_n$ with $\text{card}(\mathcal{A}_n) \leq N_{n+\tau}$ along with b_is' and $j(C)$s'. Now to obtain $\mathcal{A}_{n+1}$ we split each set of $\mathcal{A}_n$ into at most $N_{n+\tau}$ parts. Then cardinality of $\mathcal{A}_{n+1}$ will be at most $N_{n+\tau}^2 = 2^{2.2^{n+\tau}} = 2^{2^{n+\tau+1}} = N_{n+\tau+1}$. Let us take a $C \in \mathcal{A}_n$ and $j = j(C)$. We want to construct points $t_l \in C$ and sets $A_l \subset C$ for $1 \leq l \leq m = N_{n+\tau}$. First set $D_0 = C$ and choose $t_1 \in C$ such that

$$F_{n+1}\left(C \cap B\left(t_1, r^{-j-2}\right)\right) \geq \sup_{t\in C} F_{n+1}\left(C \cap B(t, r^{-j-2})\right) - \epsilon_{n+1}. \tag{6.9}$$

We then set $A_1 = C \cap B(t_1, r^{-j-1})$.

We will use induction to construct other points. Assume that we have constructed the points $t_1, \dots, t_l$ and the sets $A_1, \dots, A_l$. Set $D_l = C \setminus \cup_{p=1}^{l} A_p$. If $D_l = \phi$, then we stop the construction. If not we choose $t_{l+1} \in D_l$ such that

$$F_{n+1}\left(D_l \cap B\left(t_{l+1}, r^{-j-2}\right)\right) \geq \sup_{t \in D_l} F_{n+1}\left(D_l \cap B(t, r^{-j-2})\right) - \epsilon_{n+1}. \tag{6.10}$$

We set $A_{l+1} = D_l \cap B(t_{l+1}, r^{-j-1})$. We continue this process until we stop or up to $D_{m-1} = C \setminus \cup_{l=1}^{m-1} A_l$. If $D_{m-1} = \phi$ we stop there otherwise we set $A_m = D_{m-1}$. In this way we get a partitioned C into at most $m = N_{n+\tau}$ pieces $A_1, \dots, A_m$ along with collection of at most m points $t_1, \dots, t_m$ such that $t_i \in A_i$ for $1 \leq i < m$. Note that in the last step we have taken D_{m-1} as our A_m but we have not declared the t_m. In this case take t_C as t_m. Let A be one of $A_1, \dots, A_m$. If $A = A_m$ we define $j(A) = j = j(C)$,

$$b_0(A) = b_0(C), \quad b_1(A) = b_1(C)$$
$$b_2(A) = b_0(C) - r^{-\beta(j+1)}\theta(n+1) + \epsilon_{n+1}.$$

clearly from the definition of b_is' we can see that (6.8) is satisfied. Also observe that in this case $A = A_m \subset C \subset B(t_C, r^{-j(C)}) = B(t_m, r^{-j(A)})$. We want to prove that these choices of b_is' and j satisfy (6.4), (6.5) and (6.6). First let us show (6.6). According to our construction $t_l \in A_l \subset D_{l-1}$ for $1 \leq l \leq m$. Therefore if $l' < l$, we have $d(t_l, t_{l'}) \geq r^{-j-1}$. Hence using (6.4) for C and the growth property of F_n, used for $a = r^{-j-1}$ and $H_l = D_{l-1} \cap B(t_l, r^{-j-2})$ we get,

$$b_0(C) \geq F_n(C) \geq F_n\left(\cup_{l=1}^{m} H_l\right) \geq r^{-\beta(j+1)}\theta(n+1) + \min_{1 \leq l \leq m} F_{n+1}(H_l). \tag{6.11}$$

Here in the first inequality we are using the fact (6.4) for C and for the second inequality we are using the fact that $F_n(A) \leq F_n(A')$ if $A \subset A'$ and $\cup_{l=1}^{m} H_l \subset C$. Take any $t \in A$. Since $A \subset D_l$ for $0 \leq l \leq m-1$, therefore using (6.10) we get that

$$\begin{aligned} F_{n+1}\left(A \cap B(t, r^{-j-2})\right) &\leq F_{n+1}\left(D_l \cap B(t, r^{-j-2})\right) \\ &\leq F_{n+1}\left(D_l \cap B(t_{l+1}, r^{-j-2})\right) + \epsilon_{n+1}. \end{aligned}$$

Using (6.11) along with this relation we get,

$$\begin{aligned} F_{n+1}\left(A \cap B(t, r^{-j-2})\right) &\leq \min_{1 \leq l \leq m} F_{n+1}\left(D_{l-1} \cap B(t_l, r^{-j-2})\right) + \epsilon_{n+1} \\ &\leq b_0(C) - r^{-\beta(j+1)}\theta(n+1) + \epsilon_{n+1} \\ &= b_2(A). \end{aligned}$$

This completes the proof of (6.6). Now using the decreasing properties of F_n and (6.4) and (6.5) for C we get (6.4) and (6.5) for A. We are finished with the case $A = A_m$.

Now take $A = A_l$, where $l < m$. In this case we define $j(A) = j + 1 := j(C) + 1$ and $t_A = t_l$. Then we have

$$A = A_l \subset B(t_l, r^{-j-1}) = B(t_A, r^{-j(A)}).$$

Define,

$$\begin{aligned} b_0(A) &= b_2(A) = b_1(C) \\ b_1(A) &= \min\{b_1(C), b_2(C)\}. \end{aligned}$$

Clearly (6.8) is satisfied for the choices of b_is'. Now we want prove (6.4), (6.5), and (6.6) for A. We see that

$$\begin{aligned} F_{n+1}(A) &\leq F_{n+1}\left(C \cap B(t_l, r^{-j-1})\right) \\ &\leq F_n\left(C \cap B(t_l, r^{-j-1})\right) \\ &\leq b_1(C) = b_0(A). \end{aligned}$$

In the same way we have if $t \in A$

$$\begin{aligned} F_{n+1}\left(A \cap B(t, r^{-j(A)-1})\right) &\leq F_{n+1}\left(C \cap B(t, r^{-j-2})\right) \\ &\leq F_n\left(C \cap B(t, r^{-j-2})\right) \\ &\leq \min\left\{F_n\left(C \cap B(t, r^{-j-2})\right), F_n\left(C \cap B(t, r^{-j-1})\right)\right\} \\ &\leq \min\{b_1(C), b_2(C)\} \\ &= b_1(A). \end{aligned}$$

Also,

$$F_{n+1}\left(A \cap B(t, r^{-j(A)-2})\right) \leq F_{n+1}(A) \leq b_0(A) = b_2(A).$$

This completes the proof of (6.4), (6.5), and (6.6) for A.

We have completed the construction and move to the proof of the result stated in the theorem. First we prove a special relation among b_is'. The relation is as follows. If $n \geq 0$, $A \in \mathcal{A}_{n+1}$, $C \in \mathcal{A}_n$, and $A \subset C$ then,

$$\sum_{i=0}^{2} b_i(A) + \left(1 - \frac{1}{\zeta}\right) r^{-\beta(j(A)+1)}\theta(n+1) \leq \sum_{i=0}^{2} b_i(C) + \frac{1}{2}\left(1 - \frac{1}{\zeta}\right) r^{-\beta(j(C)+1)}\theta(n) + \epsilon_{n+1} \quad (6.12)$$

To prove this relation for our choice of $A (= A_m)$ and C, we observe that using the definition of $b_2(A)$

$$\begin{aligned} \sum_{i=0}^{2} b_i(A) + \left(1 - \frac{1}{\zeta}\right) r^{-\beta(j+1)}\theta(n+1) &= 2b_0(C) + b_1(C) - \frac{1}{\zeta} r^{-\beta(j+1)}\theta(n+1) + \epsilon_{n+1} \\ &\leq 2b_0(C) + b_1(C) - r^{-\beta(j+1)}\theta(n) + \epsilon_{n+1}. \quad (6.13) \end{aligned}$$

In the last inequality we have used the regularity condition on $\theta(n)$ i.e., $\zeta\theta(n) \leq \theta(n+1) \leq \frac{r^\beta}{2}\theta(n)$. Now using (6.8) we get that,

$$\begin{aligned}
\sum_{i=0}^{2} b_i(A) + \left(1 - \frac{1}{\zeta}\right) r^{-\beta(j(A)+1)}\theta(n+1) &\leq \sum_{i=0}^{2} b_i(C) + \epsilon_{n+1} \\
&\leq \sum_{i=0}^{2} b_i(C) + \frac{1}{2}\left(1 - \frac{1}{\zeta}\right) r^{-\beta(j(C)+1)}\theta(n) + \epsilon_{n+1}.
\end{aligned}$$

This completes the proof of (6.12) for our first choice of A i.e., A_m. For the second types of A i.e., A_l, where $l < m$ we observe that

$$\sum_{i=0}^{2} b_i(A) \leq 2b_1(C) + b_2(C) \leq \sum_{i=0}^{2} b_i(C),$$

using (6.7) for C in the last inequality. Since we have the regularity condition on $\theta(n)$ that

$$\zeta\theta(n) \leq \theta(n+1) \leq \frac{r^\beta}{2}\theta(n),$$

and $j(A) = j(C) + 1$, therefore

$$r^{-\beta(j(A)+1)}\theta(n+1) \leq \frac{1}{2} r^{-\beta(j(C)+1)}\theta(n).$$

Thus we get,

$$\sum_{i=0}^{2} b_i(A) + \left(1 - \frac{1}{\zeta}\right) r^{-\beta(j(A)+1)}\theta(n+1) \leq \sum_{i=0}^{2} b_i(C) + \frac{1}{2}\left(1 - \frac{1}{\zeta}\right) r^{-\beta(j(C)+1)}\theta(n) + \epsilon_{n+1}.$$

Now using the relation (6.12), for any $t \in T$ and any $n \geq 0$ we have (setting $j_n(t) := j(A_n(t))$)

$$\sum_{i=0}^{2} b_i(A_{n+1}(t)) + \left(1 - \frac{1}{\zeta}\right) r^{-\beta(j_{n+1}(t)+1)}\theta(n+1) \leq \sum_{i=0}^{2} b_i(A_n(t)) + \frac{1}{2}\left(1 - \frac{1}{\zeta}\right) r^{-\beta(j_n(t)+1)}\theta(n) + \epsilon_{n+1}.$$

Taking the sum over $0 \leq n \leq q$ we get,

$$\begin{aligned}
&\left(1 - \frac{1}{\zeta}\right)\sum_{n=0}^{q} r^{-\beta(j_{n+1}(t)+1)}\theta(n+1) - \frac{1}{2}\left(1 - \frac{1}{\zeta}\right)\sum_{n=0}^{q} r^{-\beta(j_n(t)+1)}\theta(n) \\
&\qquad \leq \sum_{i=0}^{2}\left(\sum_{n=0}^{q} b_i(A_n(t)) - \sum_{n=0}^{q} b_i(A_{n+1}(t))\right) + \sum_{n=0}^{q}\epsilon_{n+1}.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad &\frac{1}{2}\left(1 - \frac{1}{\zeta}\right)\sum_{n=1}^{q-1} r^{-\beta(j_n(t)+1)}\theta(n) + \left(1 - \frac{1}{\zeta}\right) r^{-\beta(j_{q+1}(t)+1)} - \frac{1}{2}\left(1 - \frac{1}{z}\right) r^{-\beta(j_0(t)+1)}\theta(0) \\
&\qquad \leq \sum_{i=0}^{2} b_i(A_0(t)) - \sum_{i=0}^{2} b_i(A_{q+1}(t)) + \sum_{n=0}^{q}\epsilon_{n+1}
\end{aligned}$$

$$\Rightarrow \quad \frac{1}{2}\left(1 - \frac{1}{\zeta}\right)\sum_{n=1}^{q-1} r^{-\beta(j_n(t)+1)}\theta(n) \leq \sum_{i=0}^{2} b_i(A_0(t)) + F_0(T) + \frac{1}{2}\left(1 - \frac{1}{\zeta}\right) r^{-\beta(j(T)+1)}\theta(0),$$

using $\sum_{n=0}^{q} \epsilon_{n+1} \leq \sum_{n=0}^{\infty} \epsilon_{n+1} \leq F_0(T)$ in the last inequality.

Taking $q \to \infty$ and observing that $b_i(A_0(t)) = F_0(T)$, $r^{-j(T)-1} < \Delta(T)$(by the definition of $j(T)$) we get

$$\sum_{n\geq 0} r^{-\beta(j_n(t)+1)}\theta(n) \leq \frac{2\zeta}{\zeta-1}4F_0(T) + \Delta^{\beta}(T)\theta(0).$$

But $\Delta(A_n(t)) \leq 2r^{-j_n(t)}$. Which gives us

$$\begin{aligned}\sum_{n\geq 0} \theta(n)\Delta^{\beta}(A_n(t)) &\leq 2^{\beta}\sum_{n\geq 0}\theta(n)r^{-\beta j_n(t)}\\ &\leq (2r)^{\beta}\sum_{n\geq 0}\theta(n)r^{-\beta(j_n(t)+1)}\\ &\leq (2r)^{\beta}\left(\frac{2\zeta}{\zeta-1}4F_0(T) + \Delta^{\beta}(T)\theta(0)\right)\\ &\leq (2r)^{\beta}\left(\frac{16F_0(T)}{\zeta-1} + \Delta^{\beta}(T)\theta(0)\right),\end{aligned}$$

using $\zeta \leq 2$ in last inequality. Taking supremum over t in the left hand side we get our result. □

Lemma 6.9. *If $f : (T,d) \to (U,d')$ is an onto function and satisfies,*

$$d'(f(x), f(y)) \leq Ad(x,y) \;\; \forall\, x, y \in T.$$

for a positive constant A, then

$$\gamma_{\alpha,\beta}(U,d') \leq K(\alpha,\beta)A\gamma_{\alpha,\beta}(T,d).$$

Proof. By the definition of $\gamma_{\alpha,\beta}(T,d)$, there exists an admissible sequence $\{\mathcal{A}_n\}$ such that

$$\sup_{t\in T}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(A_n(t),d)\right)^{\beta} \leq 2\gamma_{\alpha,\beta}^{\beta}(T,d).$$

We construct a sequence of finite sets $\{T_n\}$ such that card(T_n)=card$(\mathcal{A}_n)$ and $T_n \subset T_{n+1}$. To construct T_n, we pick one point from each set of $\mathcal{A}_n$ and take the union of it. In the next step we pick one point from those sets of $\mathcal{A}_{n+1}$ which does not intersect T_n, and add those points to T_n to get T_{n+1}. Then for any $t \in T$, we have $d(t,T_n) \leq \Delta(A_n(t),d)$. This gives us

$$\begin{aligned}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}d(t,T_n)\right)^{\beta} &\leq \sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(A_n(t),d)\right)^{\beta}\\ &\leq \sup_{t\in T}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(A_n(t),d)\right)^{\beta}\\ &\leq 2\gamma_{\alpha,\beta}^{\beta}(T,d).\\ \Rightarrow \sup_{t\in T}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}d(t,T_n)\right)^{\beta} &\leq 2\gamma_{\alpha,\beta}^{\beta}(T,d).\end{aligned}$$

Consider the sequence of finite sets $\{f(T_n)\}$. Using the above equation and the hypothesis of our theorem we have

$$\begin{aligned}\sup_{s\in U}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}d'(s,f(T_n))\right)^\beta &\leq A^\beta\sup_{t\in T}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}d(t,T_n)\right)^\beta\\ &\leq 2A^\beta\gamma_{\alpha,\beta}^\beta(T,d). \qquad (6.14)\end{aligned}$$

Now construct the voronoi cells in U using $f(T_n)$. For $u\in f(T_n)$ let

$$V(u):=\{s\in U: d'(s,f(T_n))=d'(s,u)\}.$$

We see that $\cup_{u\in f(T_n)}V(u)=U$ and for different u and u', $V(u)\cap V(u')$ shares only a part of boundaries of $V(u)$ and $V(u')$. Therefore just by modifying the boundaries of the cells a little bit we can get an admissible sequence $\mathcal{B}_n$ such that for each $B\in\mathcal{B}_n$ there exists $u\in f(T_n)$ such that $B\subset V(u)$.

Pick an $s\in U$. We see that

$$d'(s,f(T_n))\leq\Delta(B_n(s),d').$$

But to fulfil our purpose we need some kind of reverse inequality. To do so, we need to modify our admissible sequence $\{\mathcal{B}_n\}$. Consider the smallest integer $b>\frac{1}{\alpha}+1$. Pick a $B\in\mathcal{B}_n$, and define the following

$$B_{bn}:=\{s\in B: d'(s,u)\leq 2^{-bn}\Delta(U,d')\},$$

where u is that u for which $B\subset V(u)$. For $0\leq k<bn$ define the following

$$B_k=\{s\in B: 2^{-k-1}\Delta(U)<d'(s,u)\leq 2^{-k}\Delta(U)\}.$$

Basically we get a refined $B=\cup_{k=0}^{bn}B_k$ such that $\Delta(B_k,d')\leq 2^{-k+1}\Delta(U)$, and for $0\leq k<bn$

$$\Delta(B_k,d')\leq 4d'(s,f(T_n))\quad\forall\, s\in B_k.$$

Because for any $s\in B_k$, $d'(s,f(T_n))=d'(s,u)\geq 2^{-k-1}\Delta(U)\geq\frac{1}{4}\Delta(B_k,d')$. Hence

$$\Delta(B_k,d')\leq 4d'(s,f(T_n))+2^{-bn+1}\Delta(U,d')\quad\forall\, 0\leq k\leq bn,\ \ \forall\, s\in B_k. \qquad (6.15)$$

We can do this for each $B\in\mathcal{B}_n$. Let $\tilde{\mathcal{B}}_n$ be the partition of U consisting of the sets B_k, then $\mathrm{card}(\tilde{\mathcal{B}}_n)\leq(bn+1)N_n$. Define a partition $\mathcal{C}_n$ generated by $\tilde{\mathcal{B}}_0,\ldots,\tilde{\mathcal{B}}_n$. Then we have

$$\begin{aligned}\mathrm{card}(\mathcal{C}_n) &\leq \prod_{k=0}^{n}(1+bk)N_k\\ &\leq 2^{\sum_{k=0}^n 2^k}e^{b\sum_{k=0}^n k}\\ &\leq 2^{2^{n+3+b}}\\ &= N_{n+3+b},\end{aligned}$$

and $\{\mathcal{C}_n\}$ is an increasing sequence of partitions. From (6.15) we get that for each $C \in \mathcal{C}_n$

$$\Delta(C, d') \leq 4d'(s, f(T_n)) + 2^{-bn+1}\Delta(U, d') \quad \forall\, s \in C,$$

and thus using (6.14) we get that

$$\begin{aligned}
\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(C_n(s), d')\right)^\beta &\leq K(\beta)\left[\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}d'(s, f(T_n))\right)^\beta + \sum_{n\geq 0}2^{-b\beta n+\beta+\frac{n\beta}{\alpha}}\Delta^\beta(U, d')\right] \\
&\leq K(\beta)\left[2A^\beta\gamma^\beta_{\alpha,\beta}(T, d) + A^\beta\sum_{n\geq 0}2^{-n\beta+\beta}\Delta^\beta(T, d)\right] \\
&\leq K(\beta)A^\beta(2 + C(\beta))\gamma^\beta_{\alpha,\beta}(T, d),
\end{aligned}$$

where $K(\beta)$ is a constant depends only on β such that for $x, y > 0$, $(x+y)^\beta \leq K(\beta)(x^\beta + y^\beta)$, $C(\beta) = \sum_{n\geq 0} 2^{-n\beta+\beta}$, and in the last two inequalities we have used the facts that $b > \frac{1}{\alpha} + 1$ and $\Delta^\beta(U, d) \leq \gamma_{\alpha,\beta}(T, d)$. Combining all β dependent constants together we get

$$\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(C_n(s), d')\right)^\beta \leq K(\beta)A^\beta\gamma^\beta_{\alpha,\beta}(T, d).$$

The proof is almost finished here except one issue that $\{\mathcal{C}_n\}$ is not an admissible sequence, because $\mathrm{card}(\mathcal{C}_n) \leq N_{n+\beta+3} > N_n$. We can resolve this issue using the technique used in Lemma 6.8. Define an admissible sequence $\{\mathcal{D}_n\}$ by $\mathcal{D}_n = \{U\}$ if $0 \leq n < b + 3$, and $\mathcal{D}_n = \mathcal{C}_{n-b-3}$ if $n \geq b + 3$. Then

$$\begin{aligned}
\sum_{n\geq b+3}\left(2^{\frac{n}{\alpha}}\Delta(D_n(s), d')\right)^\beta &= 2^{\frac{\beta(b+3)}{\alpha}}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(C_n(s), d')\right)^\beta \\
&\leq 2^{\frac{\beta(b+3)}{\alpha}}K(\beta)A^\beta\gamma^\beta_{\alpha,\beta}(T, d) \\
&= K(\alpha, \beta)A^\beta\gamma^\beta_{\alpha,\beta}(T, d),
\end{aligned}$$

where $K(\alpha, \beta) = 2^{\frac{\beta(b+3)}{\alpha}}K(\beta)$. On the other hand, observing that $\Delta(D_n, d') = \Delta(U, d')$ for $0 \leq n < b + 3$, and $\Delta^\beta(U, d') \leq A^\beta\gamma^\beta_{\alpha,\beta}(T, d)$, we get

$$\begin{aligned}
\sum_{n=0}^{b+2}\left(2^{\frac{n}{\alpha}}\Delta(D_n(s), d')\right)^\beta &= \Delta^\beta(U, d')\sum_{n=0}^{b+2}2^{\frac{n}{\alpha}} \\
&\leq K(\alpha, \beta)A^\beta\gamma^\beta_{\alpha,\beta}(T, d),
\end{aligned}$$

where $K(\alpha, \beta) = \sum_{n=0}^{b+2} 2^{\frac{n}{\alpha}}$. Hence we get

$$\begin{aligned}
\gamma_{\alpha,\beta}(U, d') &\leq \left(\sup_{s\in U}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(D_n(s), d')\right)^\beta\right)^{\frac{1}{\beta}} \\
&\leq K(\alpha, \beta)A\gamma_{\alpha,\beta}(T, d).
\end{aligned}$$

$\square$

6.2 Ellipsoid

Definition 6.10 (Ellipsoid)**.** Let us take a sequence of complex numbers $\{a_n\}$. Define an ellipsoid as

$$\mathcal{E} = \left\{ x = (x_i) \in \mathbb{C}^{\mathbb{N}} : \sum_{i \geq 1} \left| \frac{x_i}{a_i} \right|^2 \leq 1 \right\},$$

also define a norm $\|\cdot\|_{\mathcal{E}}$ on $\mathbb{C}^{\mathbb{N}}$ as

$$\|x\|_{\mathcal{E}} = \sqrt{\sum_{i \geq 1} \left| \frac{x_i}{a_i} \right|^2}; \quad x = (x_1, x_2, \dots) \in \mathbb{C}^{\mathbb{N}}.$$

Observe that an ellipsoid $\mathcal{E}$ is the unit ball of the norm $\|\cdot\|_{\mathcal{E}}$.

Lemma 6.11. *If $\|x\|_{\mathcal{E}},\ \|y\|_{\mathcal{E}} \leq 1$, then we have*

$$\left\| \frac{x+y}{2} \right\|_{\mathcal{E}} \leq 1 - \frac{1}{8}\|x-y\|_{\mathcal{E}}^2.$$

Proof. For $x, y \in \mathcal{E}$, from parallelogram identity we get that

$$\begin{aligned}
\|x+y\|_{\mathcal{E}}^2 + \|x-y\|_{\mathcal{E}}^2 &= 2(\|x\|_{\mathcal{E}}^2 + \|y\|_{\mathcal{E}}^2) \leq 4 \\
\Rightarrow \quad \|x+y\|_{\mathcal{E}}^2 &\leq 4 - \|x-y\|_{\mathcal{E}}^2 \\
\Rightarrow \quad \left\| \frac{x+y}{2} \right\|_{\mathcal{E}}^2 &\leq 1 - \frac{1}{4}\|x-y\|_{\mathcal{E}}^2 \\
\Rightarrow \quad \left\| \frac{x+y}{2} \right\|_{\mathcal{E}} &\leq \left(1 - \frac{1}{4}\|x-y\|_{\mathcal{E}}^2\right)^{\frac{1}{2}} \\
&\leq 1 - \frac{1}{8}\|x-y\|_{\mathcal{E}}^2.
\end{aligned}$$

□

Lemma 6.12. *Take the ellipsoid $\mathcal{E}$ as our T and d is the metric induced by the ℓ^2 norm on it. Moreover, assume that the sequence $\{|a_i|\}$ is a non-increasing sequence. Then we have,*

$$e_{n+3}(\mathcal{E}, d) \leq 3 \max_{k \leq n} |a_{2^k}| 2^{k-n}.$$

Proof. Consider the following ellipsoid $\mathcal{E}_n$ in $\mathbb{C}^{2^n}$ defined as,

$$\mathcal{E}_n = \left\{ (x_i)_{i \leq 2^n} : \sum_{i=1}^{2^n} \left| \frac{x_i}{a_i} \right|^2 \leq 1 \right\}.$$

First we will prove that

$$e_{n+3}(\mathcal{E}, d) \leq e_{n+3}(\mathcal{E}_n, d) + |a_{2^n}|. \tag{6.16}$$

We observe that since $\{|a_i|\}$ is a non-increasing sequence, therefore if $x \in \mathcal{E}$ then

$$1 \geq \sum_{i \geq 1} \left| \frac{x_i}{a_i} \right|^2 \geq \frac{1}{|a_{2^n}|^2} \sum_{i > 2^n} |x_i|^2$$

which gives us $\left(\sum_{i>2^n} |x_i|^2\right)^{\frac{1}{2}} \leq |a_{2^n}|$. Take any $x = (x_i) \in \mathcal{E}$ and define $\tilde{x} = (x_1, \ldots, x_{2^n}) \in \mathcal{E}_n$. So we can think the elements of $\mathcal{E}_n$ as the finite dimensional projections of the vectors of $\mathcal{E}$, and we can view $\mathcal{E}_n$ as the subset of $\mathcal{E}$. Now for any $x \in \mathcal{E}$ we have,

$$d(x, \tilde{x}) = \left(\sum_{i > 2^n} |x_i|^2 \right)^{\frac{1}{2}} \leq |a_{2^n}|.$$

If we take any finite subset $T_n \subset \mathcal{E}_n$ such that $\operatorname{card}(T_n) \leq N_{n+3}$ then in particular T_n is also a finite subset of $\mathcal{E}$ satisfying $\operatorname{card}(T_n) \leq N_{n+3}$. Now for any $x \in \mathcal{E}$ we have,

$$\begin{aligned} d(x, T_n) &\leq d(x, \tilde{x}) + d(\tilde{x}, T_n) \\ &\leq |a_{2^n}| + d(\tilde{x}, T_n) \\ &\leq |a_{2^n}| + \sup_{y \in \mathcal{E}_n} d(y, T_n) \\ \Rightarrow \sup_{x \in \mathcal{E}} d(x, T_n) &\leq |a_{2^n}| + \sup_{y \in \mathcal{E}_n} d(y, T_n). \end{aligned}$$

We observe that here T_ns' are particular type of subsets of $\mathcal{E}$. Therefore

$$\begin{aligned} e_{n+3}(\mathcal{E}, d) &\leq \sup_{x \in \mathcal{E}} d(x, T_n) \\ &\leq |a_{2^n}| + \sup_{y \in \mathcal{E}_n} d(y, T_n). \end{aligned}$$

Now taking infimum over all possible such T_ns' in the right hand side, we get

$$e_{n+3}(\mathcal{E}, d) \leq |a_{2^n}| + e_{n+3}(\mathcal{E}_n, d).$$

Take an $\epsilon > 0$ and construct a subset Z of $\mathcal{E}_n$ such that $\operatorname{card}(Z)$ is the as large as possible and any two points of Z are at least 2ϵ distance apart from each other. Observe that the collection of balls of radius 2ϵ centered at the points of Z covers the whole $\mathcal{E}_n$. If not then there would exist a point $x \in \mathcal{E}_n$ which is not covered by the union of above balls and which would imply that x is at least 2ϵ distance apart from any point of Z. So we can include x and construct another subset $Z' = Z \cup \{x\}$ of $\mathcal{E}_n$ such that any two points of Z' is at least 2ϵ distance apart from each other. That would contradict the maximal cardinality of Z.

From our construction of Z we see that $d(x, Z) \leq 2\epsilon$ for all $x \in \mathcal{E}$, therefore if $\operatorname{card}(Z) \leq N_{n+3}$ then,

$$e_{n+3}(\mathcal{E}_n, d) \leq 2\epsilon. \tag{6.17}$$

Also from the construction of Z we observe that the balls of radius ϵ centered at the points of Z have disjoint interiors. Therefore if B is the centered unit ball of $\mathbb{C}^{2^n}$ with euclidian distance, then

$$\mathrm{card}(Z)\mathrm{vol}(\epsilon B) \le \mathrm{vol}(\mathcal{E}_n + \epsilon B). \tag{6.18}$$

Now for $x = (x_1, \ldots, x_{2^n}) \in \mathcal{E}_n$ we have $\sum_{i=1}^{2^n} \left|\frac{x_i}{a_i}\right|^2 \le 1$, and for $x' = (x'_1, \ldots, x'_{2^n}) \in \epsilon B$ we have $\sum_{i=1}^{2^n} |x'_i|^2 \le \epsilon^2$. Since $|x_i + x'_i|^2 \le 2(|x_i|^2 + |x'_i|^2)$, we have

$$\mathcal{E}_n + \epsilon B \subset \mathcal{E}^1 = \left\{ x = (x_i)_{i \le 2^n} : \sum_{i=1}^{2^n} \left|\frac{x_i}{c_i}\right|^2 \le 1 \right\},$$

where $c_i = 2\max\{\epsilon, |a_i|\}$. Thus we get

$$\mathrm{vol}(\mathcal{E}_n + \epsilon B) \le \mathrm{vol}(\mathcal{E}^1) = \mathrm{vol}(B) \prod_{i=1}^{2^n} c_i.$$

Combining this with equation (6.18) we get

$$\mathrm{card}(Z) \le \prod_{i=1}^{2^n} \frac{c_i}{\epsilon} = 2^{2^n} \prod_{i=1}^{2^n} \max\left\{1, \frac{|a_i|}{\epsilon}\right\}.$$

Let $\epsilon = \max_{k \le n} |a_k| 2^{k-n}$, then for $k \le n$, $|a_{2^k}| \le \epsilon 2^{n-k}$. Since $\{|a_i|\}$ is a non-increasing therefore $|a_i| \le |a_{2^k}| \le \epsilon 2^{n-k}$ for $2^k < i \le 2^{k+1}$. So we get

$$\begin{aligned}
\prod_{i=1}^{2^n} \max\left\{1, \frac{|a_i|}{\epsilon}\right\} &= \left(\prod_{k=0}^{n-1} \prod_{i=2^k}^{2^{k+1}-1} \max\left\{1, \frac{|a_i|}{\epsilon}\right\} \right) \max\left\{1, \frac{|a_{2^n}|}{\epsilon}\right\} \\
&\le \prod_{k=0}^{n-1} 2^{(n-k)2^k} \\
&= 2^{\sum_{k=0}^{n-1} (n-k)2^k} \\
&= 2^{\sum_{i=1}^{n} i 2^{n-i}} \\
&= 2^{2^n \sum_{i=1}^{n} i 2^{-i}} \\
&\le 2^{2^{n+2}},
\end{aligned}$$

using the fact $\sum_{i=1}^{\infty} i 2^{-i} \le 4$ in the last inequality. Thus $\mathrm{card}(Z) \le 2^{2^n} \cdot 2^{2^{n+2}} \le 2^{2^{n+3}}$. Hence $e_{n+3}(\mathcal{E}_n, d) \le 2\max_{k \le n} |a_{2^k}| 2^{k-n}$, and using (6.16) we get our desired result that $e_{n+3}(\mathcal{E}, d) \le \max_{k \le n} |a_{2^k}| 2^{k-n} + |a_{2^n}| \le 3 \max_{k \le n} |a_{2^k}| 2^{k-n}$. □

Definition 6.13. Consider a number $p \ge 2$. A norm $\|\cdot\|$ in a Banach space is called $p-$convex if for a certain number $\eta > 0$ we have,

$$\|x\|,\ \|y\| \le 1 \quad \Rightarrow \quad \left\| \frac{x+y}{2} \right\| \le 1 - \eta \|x - y\|^p. \tag{6.19}$$

For example from the Lemma 6.11 we see that the ellipsoid norm is $2-$convex. Here we will deal with metric spaces (T,d), where T is the unit ball of a p-convex Banach Space B, and d is the distance induced on B by another norm $\|\cdot\|'$.

Given any metric space (T,d) recall the functionals,

$$\gamma_{\alpha,\beta}(T,d) = \left(\inf \sup_{t\in T} \sum_{n\geq 0} \left(2^{\frac{n}{\alpha}} \Delta(A_n(t))\right)^{\beta}\right)^{\frac{1}{\beta}},$$

where infimum is taken over all admissible sequences.

Theorem 6.14. *Let T be the unit ball of a p-convex Banach space, η be same as in* (6.19), *and d be a distance on T induced by another norm $\|\cdot\|'$. Consider a sequence $\{\theta(n)\}$ such that*

$$\theta(n) \leq \eta \left(\frac{1}{8e_n(T,d)}\right)^p \quad \forall\, n \geq 0, \tag{6.20}$$

and for certain numbers $1 < \zeta \leq 2$, $r \geq 4$

$$\zeta\theta(n) \leq \theta(n+1) \leq \frac{r^p}{2}\theta(n). \tag{6.21}$$

Then there exists an increasing sequence $\{\mathcal{A}_n\}$ of partitions of T satisfying card$(\mathcal{A}_n) \leq N_{n+1} = 2^{2^{n+1}}$ such that

$$\sup_{t\in T} \sum_{n\geq 0} \theta(n)\Delta(A_n(t),d)^p \leq L\frac{(2r)^p}{\zeta - 1}.$$

Proof. Observe that this is kind of a special case of Theorem 6.7. Here we are in a setting of an ellipsoid. We will use the Theorem 6.7 with $\tau = 1$ and $\beta = p$. We take the functionals F_n as

$$F_n(A) = F(A) := 1 - \inf\{\|v\| : v \in \text{conv}(A)\},$$

where conv(A) means the convex hull generated by A, and $\|\cdot\|$ is the ellipsoid norm $\|\cdot\|_{\mathcal{E}}$. To use the Theorem 6.7 we need to show that these functionals satisfy the growth condition of the Definition 6.6. Consider $a > 0$, $n \geq 0$, $m = N_{n+1}$, and points $\{t_l\}_{l\leq m}$ in T such that $d(t_l, t_{l'}) \geq a$ whenever $l \neq l'$. Also consider the sets $H_l \subset T \cap B_d(t_l, \frac{a}{r})$, where the suffix d indicates that the ball is with respect to the distance d rather than the ellipsoid norm $\|\cdot\| =$.

Define

$$u := \inf\left\{\|v\| : v \in \text{conv}\left(\cup_{l=1}^m H_l\right)\right\} = 1 - F\left(\cup_{l=1}^m H_l\right), \tag{6.22}$$

and we take

$$u' > \max_{1\leq l\leq m} \inf\{\|v\| : v \in \text{conv}(H_l)\} = 1 - \min_{1\leq l\leq m} F(H_l) \tag{6.23}$$

Take points $v_l \in H_l$ for $1 \leq l \leq m$ such that $\|v_l\| \leq u'' := \min\{u', 1\}$. From the definition of p-convexity (6.19) it follows that for $l, l' \leq m$,

$$\left\| \frac{v_l + v_{l'}}{2u''} \right\| \leq 1 - \eta \left\| \frac{v_l - v_{l'}}{u''} \right\|^p. \tag{6.24}$$

Also we observe that $\frac{v_l + v_{l'}}{2} \in \operatorname{conv}(\cup_{l=1}^m H_l)$, therefore $u \leq \left\| \frac{v_l + v_{l'}}{2} \right\|$, and hence from (6.24) we get

$$\frac{u}{u''} \leq 1 - \eta \left\| \frac{v_l - v_{l'}}{u''} \right\|^p.$$

It implies that

$$\|v_l - v_{l'}\| \leq u'' \left(\frac{u'' - u}{\eta u''} \right)^{\frac{1}{p}} \leq R := \left(\frac{u'' - u}{\eta} \right)^{\frac{1}{p}},$$

since $u'' \leq 1$ and $p \geq 2$. Consider the points $w_l = \frac{v_l - v_1}{R}$, then from the above relationship we get that $\|w_l\| \leq 1$ and hence $w_l \in T$. Now, since $v_l \in H_l \subset T \cap B_d(t_l, \frac{a}{r})$ and $d(t_l, t_{l'}) \geq a$ for $l \neq l'$, therefore $d(v_l, v_{l'}) \geq a - \frac{2a}{r} \geq \frac{a}{2}$ (since $r \geq 4$) for $l \neq l'$, and hence $d(w_l, w_{l'}) \geq \frac{a}{2R}$ (since d arises from a norm) for $l \neq l'$.

Thus we get a collection of $m = N_{n+1}$ points $w_1, \ldots, w_m$ in T such that $d(w_l, w_{l'}) \geq \frac{a}{2R}$ for $l \neq l'$. We claim that if $S = \{s_1, \ldots, s_m\}$ is another finite subset of T, then $\sup_{t \in T} d(t, S) \geq \frac{a}{8R}$. If not, then $d(t, S) < \frac{a}{8R}$ for all $t \in T$. This gives us that $\mathscr{C} = \cup_{i=1}^m B_d(s_i, \frac{a}{8R})$, the union of balls of radius $\frac{a}{8R}$ around $s_1, \ldots, s_m$ covers the whole T. In particular, $\{w_1, \ldots, w_m\} \in \mathscr{C}$. But for $l \neq l'$, w_l and $w_{l'}$ cannot stay in the same ball because $d(w_l, w_{l'}) \geq \frac{a}{2R}$ for $l \neq l'$. Therefore each $B_d(s_i, \frac{a}{8R})$ contains only one w_i. Without loss of generality let us say $w_i \in B_d(s_i, \frac{a}{8R})$. Now take the union of balls of radius $\frac{a}{4R}$ around w_is', say $\mathscr{D} = \cup_{i=1}^m B_d(w_i, \frac{a}{4R})$. Since $d(w_l, w_{l'}) \geq \frac{a}{2R}$ for $l \neq l'$, therefore $\mathscr{D}$ cannot cover the whole T. Take $s \in T$ such that $s \notin \mathscr{D}$. But since $\mathscr{C}$ covers T, so there exists $1 \leq i \leq m$ such that $s \in B_d(s_i, \frac{a}{8R})$. Now since $w_i \in B_d(s_i, \frac{a}{8R})$, we have $d(s, w_i) < \frac{a}{4R}$, which contradicts the fact that $s \notin \mathscr{D}$.

Thus we get $e_{n+1}(T, d) \geq \frac{a}{8R}$.

Observing that $u' - u \geq u'' - u = \eta R^p$, we have

$$u' \geq u + \eta \left(\frac{a}{8e_{n+1}(T, d)} \right)^p.$$

But u' is arbitrary in (6.23), therefore we get

$$\begin{aligned}
1 - \min_{1 \leq l \leq m} F(H_l) &\geq u + \eta \left(\frac{a}{8e_{n+1}(T, d)} \right)^p \\
\Rightarrow \quad 1 - u &\geq \min_{1 \leq l \leq m} F(H_l) + a^p \theta(n+1) \\
\Rightarrow F\left(\cup_{l=1}^m H_l\right) &\geq \min_{1 \leq l \leq m} F(H_l) + a^p \theta(n+1).
\end{aligned}$$

This shows that F_ns' satisfy the growth condition of the Definition 6.6.

We observe that $F_0(T) = F(T) = 1$ and $\Delta(T,d) \le 2e_0(T,d)$, therefore using (6.20) we get

$$\begin{aligned} \theta(0)\Delta(T,d)^p &\le \theta(0)2^p e_0^p(T,d) \\ &\le \eta 2^{-2p} \\ &\le 1, \end{aligned}$$

where the last inequality can be easily derived from (6.19). Thus using the Theorem 6.7 we can say that there exists an increasing sequence $\{\mathcal{A}_n\}$ of partitions of T satisfying $\operatorname{card}(\mathcal{A}_n) \le N_{n+1}$ such that

$$\begin{aligned} \sup_{t\in T}\sum_{n\ge 0}\theta(n)\Delta(A_n(t),d)^p &\le L(2r)^p\left(\frac{F_0(T)}{\zeta-1}+\theta(0)\Delta(T,d)^p\right) \\ &\le L\frac{(2r)^p}{\zeta-1}+L(2r)^p \\ &\le L'\frac{(2r)^p}{\zeta-1} \quad (\text{where } L' = 2L). \end{aligned}$$

□

As a consequence of this theorem we get the following.

Theorem 6.15. *If T is the unit ball of a p-convex Banach space, if η is as in* (6.19) *and if the distance d on T is induced by another norm $\|\cdot\|'$, then for $\alpha \ge 1$ we have*

$$\gamma_{\alpha,p}(T,d) \le K(\alpha,p,\eta)\sup_{n\ge 0} 2^{\frac{n}{\alpha}} e_n(T,d).$$

Proof. Define

$$S = \sup_{n\ge 0} 2^{\frac{n}{\alpha}} e_n(T,d),$$

and take $\theta(n) = \eta\frac{2^{\frac{np}{\alpha}}}{(8S)^p}$ in the Theorem 6.14. It is easy to check that $\theta(n)$ satisfies (6.20) and (6.21) for $\zeta = \min\{2, 2^{\frac{p}{\alpha}}\}$, and $r = \max\{4, 2^{\frac{1}{p}+\frac{1}{\alpha}}\}$. Then we see that $1 < \zeta \le 2$ and $r \ge 4$. Now we construct an admissible sequence $\{\mathcal{B}_n\}$ by taking $\mathcal{B}_0 = \{T\}$ and $\mathcal{B}_n = \mathcal{A}_{n-1}$ for $n \ge 1$. Then using Theorem 6.14 we have

$$\begin{aligned} \frac{\eta}{(8S)^p}\gamma_{\alpha,p}^p(T,d) &\le \sup_{t\in T}\sum_{n\ge 0}\theta(n)\Delta(B_n(t),d)^p \\ &= \sup_{t\in T}\sum_{n\ge 0}\theta(n+1)\Delta(A_n(t),d)^p + \theta(0)\Delta(B_0(t),d)^p \\ &\le \sup_{t\in T}\sum_{n\ge 0}2^{\frac{p}{\alpha}}\theta(n)\Delta(A_n(t),d)^p + 1 \\ &\le L\frac{(2r)^p}{\zeta-1}2^{\frac{p}{\alpha}} + 1 \\ &:= K(\alpha,p). \end{aligned}$$

Define $K(\alpha,p,\eta)=8\left(\frac{K(\alpha,p)}{\eta}\right)^{\frac{1}{p}}$, then we get

$$\gamma_{\alpha,p}(T,d) \quad \leq \quad K(\alpha,p,\eta)\sup_{n\geq 0} 2^{\frac{n}{\alpha}} e_n(T,d).$$

□

We observe another easy fact that

$$\sup_{n\geq 0} 2^{\frac{n}{\alpha}} e_n(T,d) \leq K(\alpha,\beta)\gamma_{\alpha,\beta}(T,d).$$

This can be proved in the following way.

By the definition of $\gamma_{\alpha,\beta}$, there exists an admissible sequence $\{\mathcal{A}_n\}$ such that

$$\sup_{t\in T}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(A_n(t),t)\right)^{\beta} \leq 2\gamma^{\beta}_{\alpha,\beta}(T,d).$$

Corresponding to that admissible sequence we can construct a sequence of finite subsets $\{T_n\}$ of T such that $\text{card}(T_n)\leq N_n$ just by picking one point from each set of $\mathcal{A}_n$ and taking collection of it. Take any $t\in T$ then $d(t,T_n)\leq \Delta(A_n(t),d)$ and therefore it is easy to see that

$$\begin{aligned}
\left(2^{\frac{n}{\alpha}}\sup_{t\in T} d(t,T_n)\right)^{\beta} &\leq \sup_{t\in T}\sum_{n\geq 0}\left(2^{\frac{n}{\alpha}}\Delta(A_n(t),d)\right)^{\beta} \leq 2\gamma^{\beta}_{\alpha,\beta}(T,d)\\
\Rightarrow \quad 2^{\frac{n}{\alpha}} e_n(T,d) &\leq 2^{\frac{1}{\beta}}\gamma_{\alpha,\beta}(T,d)\\
\Rightarrow \quad \sup_{n\geq 0} 2^{\frac{n}{\alpha}} e_n(T,d) &\leq K(\alpha,\beta)\gamma_{\alpha,\beta}(T,d).
\end{aligned}$$

As a corollary of the previous theorem we have the following

Corollary 6.16 (The Ellipsoid Theorem)**.** *Consider the ellipsoid in Definition 6.10, where the numbers $|a_i|$ are in non-increasing order. Let d be the metric induced by ℓ^2 norm on it. Then we have*

$$\gamma_{1,2}(\mathcal{E},d) \leq L\sup_{\epsilon\geq 0}\epsilon\left(card\{i : |a_i|\geq \epsilon\}\right).$$

Proof. Using the theorem 6.15 with $\|\cdot\|'$ as ℓ^2 norm we get that

$$\gamma_{1,2}(\mathcal{E},d)\leq L\sup_{n\geq 0} 2^n e_n(\mathcal{E}).$$

Now using Lemma 6.12 we get that

$$\gamma_{1,2}(\mathcal{E},d)\leq L\sup_{n\geq 0} 2^n\left(\max_{0\leq k\leq n} 2^{k-n}|a_{2^k}|\right).$$

Fix an $n\geq 0$, and let $\max_{0\leq k\leq n} 2^{k-n}|a_{2^k}| = 2^{l-n}|a_{2^l}|$ for some $0\leq l\leq n$. Then we have

$$2^n\max_{0\leq k\leq n} 2^{k-n}|a_{2^k}| = 2^l|a_{2^l}|.$$

Since $\{|a_i|\}$ is a non-increasing sequence, therefore we have $\text{card}\{i : |a_i| \geq |a_{2^l}|\} = 2^l + 1$. Thus taking $\epsilon = |a_{2^l}|$, we get

$$\begin{aligned} 2^l |a_{2^l}| &\leq \epsilon\left(\text{card}\{i : |a_i| \geq \epsilon\}\right) \\ &\leq \sup_{\epsilon>0} \epsilon\left(\text{card}\{i : |a_i| \geq \epsilon\}\right). \end{aligned}$$

Hence we get the result

$$\gamma_{1,2}(\mathcal{E}, d) \leq L \sup_{\epsilon>0} \epsilon\left(\text{card}\{i : |a_i| \geq \epsilon\}\right).$$

□

6.3 Proof of Lemma 5.4:

Now we move to our matching problem. Recall that we need to prove Lemma 5.4 in the previous chapter. We will make use of the Generic chaining method here. Before moving to the proof of the Lemma 5.4 we need some results.

Lemma 6.17. *For a fixed w and fixed k, card$(C(w,k)) \leq N_{k+l_1+1}$.*

Proof. Any curve $C \in C(w,k)$ has length at most 2^k which means that it consists of at most 2^{k+l_1} many edges. Starting from the vertex w at each step we can move in four directions and hence the total number of possible curves in $C(w,k)$ is bounded by $4^{2^{k+l_1}} = 2^{2^{k+l_1+1}} = N_{k+l_1+1}$. □

Lemma 6.18. *Consider a metric space (T,d) with card$(T) \leq N_m$. Then*

$$\gamma_2(T, \sqrt{d}) \leq 2^{\frac{1}{4}} m^{\frac{3}{4}} \gamma_{1,2}(T,d)^{\frac{1}{2}}.$$

Proof. From the definition of $\gamma_{1,2}$ we can get an admissible sequence $\{\mathcal{A}_n\}$ of T such that,

$$\forall\, t \in T, \quad \sum_{n\geq 0} \left(2^n \Delta(A_n(t), d)\right)^2 \leq 2\gamma_{1,2}^2(T,d). \tag{6.25}$$

Without loss of generality we can assume that card$(\mathcal{A}_m)$=card(T) i.e., $A_m(t) = \{t\}$ for each $t \in T$ (since we know that card(T), card$(\mathcal{A}_m) \leq N_m$ and inserting more sets into a partition will make it finer i.e., making $\Delta(A_n(t), d)$ smaller and therefore (6.25) will still remain true). In that case the sum in the left hand side of (6.25) is actually over $n \leq m-1$. Since $\sqrt{x}$ is an increasing function,

$$\begin{aligned} \sqrt{\sup_{s,t\in A} d(s,t)} &= \sup_{s,t\in A} \sqrt{d(s,t)} \\ \text{i.e.,} \quad \sqrt{\Delta(A,d)} &= \Delta(A, \sqrt{d}. \end{aligned}$$

Therefore using Hölder's inequality we get that

$$\begin{aligned}\sum_{0\le m-1} 2^{\frac{n}{2}}\Delta(A_n(t),\sqrt{d}) &= \sum_{0\le m-1}(2^n\Delta(A_n(t),d))^{\frac{1}{2}}\\ &\le m^{\frac{3}{4}}\left[\sum_{0\le m-1}(2^n\Delta(A_n(t),d))^2\right]^{\frac{1}{4}}\\ &\le 2^{\frac{1}{4}}m^{\frac{3}{4}}\gamma_{1,2}(T,d)^{\frac{1}{2}}.\end{aligned}$$

□

Lemma 6.19. *Consider the set $\mathcal{L}$ of functions $f:[0,1]\to\mathbb{R}$ such that $f(0)=f(\frac{1}{2})=f(1)=0$, f is continuous on $[0,1]$, f is differentiable outside a finite set and $\sup|f'|\le 1$. Then $\gamma_{1,2}(\mathcal{L},d_2)\le L$, where $d_2(f,g)=\|f-g\|_2=\left(\int_0^1(f(x)-g(x))^2\,dx\right)^{\frac{1}{2}}$ and L is a universal constant.*

Proof. Using Fourier transform we get,

$$\|f\|_2^2=\sum_{p\in\mathbb{Z}}|C_p(f)|^2,$$

where,

$$C_p(f)=\int_0^1 e^{i2\pi px}f(x)\,dx \quad p\in\mathbb{Z}.$$

We see that,

$$\begin{aligned}0 &= \int_0^1\frac{d}{dx}\left(e^{i2\pi px}f(x)\right)\,dx\\ &= i2\pi p\int_0^1 e^{i2\pi px}f(x)\,dx+\int_0^1 e^{i2\pi px}f'(x)\,dx\\ &= i2\pi pC_p(f)+C_p(f')\\ \Rightarrow\quad C_p(f') &= -i2\pi pC_p(f).\end{aligned}$$

Therefore $4\pi^2\sum_{p\in\mathbb{Z}}p^2|C_p(f)|^2=\sum_{p\in\mathbb{Z}}|C_p(f')|^2=\|f'\|_2^2\le 1$. Which gives us $\sum_{p\in\mathbb{Z}}p^2|C_p(f)|^2\le\frac{1}{4\pi^2}$. Also we have the following,

$$\begin{aligned}|C_0(f)|^2 &= \left|\int_0^1 f(x)\,dx\right|^2\\ &\le \int_0^1|f(x)|^2\,dx\\ &= \|f\|_2^2\\ \Rightarrow\quad |C_0(f)| &\le \|f\|_2.\end{aligned}$$

Define
$$\mathcal{E} = \left\{ (C_p) \in l^2_{\mathbb{C}}(\mathbb{Z}) : \sum_{p \in \mathbb{Z}} p^2 |C_p|^2 \leq 1 \right\}.$$
We observe that for $0 < \epsilon \leq 1$,
$$\text{card}\left\{ p \in \mathbb{Z}\backslash\{0\} : |p| < \frac{1}{\epsilon} \right\} \leq \frac{2}{\epsilon} + 1 \leq \frac{3}{\epsilon},$$
and for $\epsilon > 1$
$$\text{card}\left\{ p \in \mathbb{Z}\backslash\{0\} : |p| < \frac{1}{\epsilon} \right\} = 0$$
. Hence
$$\sup_{\epsilon > 0} \epsilon \left(\text{card}\left\{ p \in \mathbb{Z}\backslash\{0\} : \frac{1}{|p|} > \epsilon \right\} \right) \leq 3.$$
So using corollary 6.16 we have,
$$\gamma_{1,2}(\mathcal{E}) \leq 3L' = L,$$
where L' is a universal constant in Corollary 6.16. Now we see that $\mathcal{L}$ is isometrically embedded in $\mathcal{E}$. Therefore using the lemma 6.9 we get our desired result that,
$$\gamma_{1,2}(\mathcal{L}, d_2) \leq L.$$

□

On the set of simple closed curves traced on G, define the metric d_1 as
$$d_1(C, C') = \lambda(\mathring{C} \Delta \mathring{C}'). \tag{6.26}$$
The following proposition states about the metric space $C(w, k)$ with the metric d_1. This proposition will be main part of proving Lemma 5.4.

Proposition 6.20. *Consider the metric space $(C(w, k), d_1)$, then we have the following*
$$\gamma_{1,2}(C(w, k), d_1) \leq L2^{2k}.$$

Proof. Recall that in the previous lemma we defined $\mathcal{L}$ as the set of continuous functions $f : [0, 1] \to \mathbb{R}$ such that $f(0) = f(\frac{1}{2}) = f(1) = 0$, and f is differentiable everywhere except finitely many points satisfying the property that $|f'| \leq 1$. With each such function we can associate a simple closed curve $W(f)$ defined by,
$$u \mapsto \quad \left(w_1 + 2^{k+1} f\left(\frac{u}{2}\right),\ w_2 + 2^{k+1} f\left(\frac{u+1}{2}\right) \right),$$
where $(w_1, w_2) = w$ refers to the w of $C(w, k)$ (note that, to obey $|f'| \leq 1$, we have taken $2^{k+1} f\left(\frac{u}{2}\right)$ instead of just $f\left(\frac{u}{2}\right)$). We see that not all such curves trace on G. But for any

curve $C \in C(w,k)$ we can find a function f such that the above defined curve is actually C. So we can say that $C(w,k) \subset W(\mathcal{L})$. Consider $T = W^{-1}(C(w,k))$. Take $f_0, f_1 \in T$ and define the map $h : [0,1]^2 \to [0,1]^2$ given by,

$$h(u,v) = \left(w_1 + 2^{k+1}\left(vf_0\left(\frac{u}{2}\right) + (1-v)f_1\left(\frac{u}{2}\right)\right),\ w_2 + 2^{k+1}\left(vf_0\left(\frac{u+1}{2}\right) + (1-v)f_1\left(\frac{u+1}{2}\right)\right)\right).$$

Observe that h is kind of a convex combination of two curves rooted at $w = (w_1, w_2)$. So via h we can get an estimate of $\lambda(\mathring{W}(f_0), \mathring{W}(f_1)) = d_1(W(f_0), W(f_1))$. Take $x \notin h([0,1]^2)$. Now since $W(f_0)$ and $W(f_1)$ are homotopic to each other via the map h and $x \notin h([0,1]^2)$. Therefore both the curves $W(f_0)$ and $W(f_1)$ turn the same number of times around x. So either $x \in \mathring{W}(f_0) \cap \mathring{W}(f_1)$ (in case that the winding number of $W(f_0)$ & $W(f_1)$ around x is one) or $x \notin \mathring{W}(f_0) \cup \mathring{W}(f_1)$ (in case that the winding number of $W(f_0)$ & $W(f_1)$ around x is zero). Thus we get $\mathring{W}(f_0)\Delta\mathring{W}(f_1) \subset h([0,1]^2)$, which gives us

$$d_1\left(W(f_0), W(f_1)\right) \leq \lambda\left(h([0,1]^2)\right).$$

To estimate $d_1(W(f_0), W(f_1))$ we need to find out the area of $h([0,1]^2)$. But the area of $h([0,1]^2)$ is $\int\int_{[0,1]^2} |Jh(u,v)|\,dudv$ where Jh is the jacobian of h. since $|f_0'| \leq 1$ and $|f_1' \leq 1|$ therefore we see that,

$$\begin{aligned}
|Jh(u,v)| &= 2^{2k+2}\left|\left(\frac{v}{2}f_0'\left(\frac{u}{2}\right) + \frac{1-v}{2}f_1'\left(\frac{u}{2}\right)\right)\left(f_0\left(\frac{u+1}{2}\right) - f_1\left(\frac{u+1}{2}\right)\right)\right. \\
&- \left.\left(f_0\left(\frac{u}{2}\right) - f_1\left(\frac{u}{2}\right)\right)\left(\frac{v}{2}f_0'\left(\frac{u+1}{2}\right) + \frac{1-v}{2}f_1'\left(\frac{u+1}{2}\right)\right)\right| \\
&\leq 2^{2k+1}\left(\left|f_0\left(\frac{u+1}{2}\right) - f_1\left(\frac{u+1}{2}\right)\right| + \left|f_0\left(\frac{u}{2}\right) - f_1\left(\frac{u}{2}\right)\right|\right).
\end{aligned}$$

Hence

$$\left(\int\int_{[0,1]^2} |Jh(u,v)|\ dudv\right)^2 \leq \int\int_{[0,1]^2} |Jh(u,v)|^2\ dudv \leq L2^{4k}\|f_0 - f_1\|_2^2.$$

Which gives us

$$d_1\left(W(f_0), W(f_1)\right) \leq L2^{2k}\|f_0 - f_1\|_2.$$

Using the Lemma 6.9 and Lemma 6.19 we get our desired result that

$$\gamma_{1,2}\left(C(w,k), d_1\right) \leq L'2^{2k}\gamma_{1,2}(T, d_2) \leq L2^{2k}.$$

□

We move to the proof of Lemma 5.4. To prove this proposition we will use the Theorem 6.4. On the set of simple closed curves consider another metric d_2 defined as,

$$d_2(C_1, C_2) := \sqrt{n}\left\|\mathbf{1}_{\mathring{C}_1} - \mathbf{1}_{\mathring{C}_2}\right\|_2 = \left(nd_1(C_1, C_2)\right)^{\frac{1}{2}}. \tag{6.27}$$

Then we have

$$\gamma_2\left(C(w,k),d_2\right) \le \sqrt{n}\gamma_2\left(C(w,k),\sqrt{d_1}\right).$$

When $k \le l_1+2$ we have $m := k+l_1+1 \le 2l_1+3 \le L\log n$. Using Lemma 6.18 and Proposition 6.20 we get,

$$\begin{aligned}\gamma_2\left(C(w,k),d_2\right) &\le L'\sqrt{n}(\log n)^{\frac{3}{4}}\gamma_{1,2}^{\frac{1}{2}}\left(C(w,k),d_1\right)\\ &\le L2^k\sqrt{n}(\log n)^{\frac{3}{4}}. \end{aligned} \tag{6.28}$$

In Theorem 6.4 take $T = C(w,k)$, and we define the process $X_C = \frac{1}{L}\sum_{i=1}^n \left(\mathbf{1}_{\mathring{C}}(X_i) - \lambda(\mathring{C})\right)$; $C \in T$, where L is a constant to be announced later. Take two metrics δ and d_2 on T, where $\delta(C,C') = \mathbf{1}_{\{C\neq C'\}}$ and d_2 is same as above i.e., $d_2(C,C') = \sqrt{n}\lambda(\mathring{C}\Delta\mathring{C}')^{\frac{1}{2}} = (nd_1(C,C'))^{\frac{1}{2}}$. We want to show this stochastic process satisfies the hypothesis of Theorem 6.4. To show this we use the Bernstein's inequality.

Bernstein's Inequality: Let Z_is' be a collection of independent random variables such that $\mathbb{E}[Z_i] = 0$ for all i, and $|Z_i| \le M$ for all i for some $M > 0$. Define, $S_n = \sum_{i=1}^n Z_i$ then for any $u > 0$ the following holds true

$$\mathbb{P}(S_n > u) \le \exp\left\{-\frac{u^2/2}{\sum_{i=1}^n \mathbb{E}[Z_i^2] + Mu/3}\right\}.$$

We observe that the random variable

$$X_C - X_{C'} = \frac{1}{L}\sum_{i=1}^n\left(\mathbf{1}_{\mathring{C}}(X_i) - \mathbf{1}_{\mathring{C}'}(X_i) - \lambda(\mathring{C}) + \lambda(\mathring{C}')\right)$$

can be written as the sum of n independent random variables Z_i, where

$$LZ_i = \begin{cases} 1-\lambda(\mathring{C})+\lambda(\mathring{C}') & \text{w.p. } \lambda(\mathring{C}\backslash\mathring{C}') \\ -\lambda(\mathring{C})+\lambda(\mathring{C}') & \text{w.p. } 1-\lambda(\mathring{C}\Delta\mathring{C}') \\ -1-\lambda(\mathring{C})+\lambda(\mathring{C}') & \text{w.p. } \lambda(\mathring{C}'\backslash\mathring{C}) \end{cases}$$

We see that

$$\begin{aligned} L\mathbb{E}[Z_i] &= \left[\lambda(\mathring{C}\backslash\mathring{C}') - \lambda(\mathring{C}'\backslash\mathring{C})\right] + \lambda(\mathring{C}') - \lambda(\mathring{C}) \\ &\quad + \left(\lambda(\mathring{C}') - \lambda(\mathring{C})\right)\times\left[\lambda(\mathring{C}\backslash\mathring{C}') + \lambda(\mathring{C}'\backslash\mathring{C}) - \lambda(\mathring{C}\Delta\mathring{C}')\right] \\ &= [\lambda(\mathring{C}) - \lambda(\mathring{C}')] + [\lambda(\mathring{C}') - \lambda(\mathring{C})] \\ &= 0, \end{aligned}$$

$$\begin{aligned} L^2\mathbb{E}[Z_i^2] &= \left[\lambda(\mathring{C}\backslash\mathring{C}') + \lambda(\mathring{C}'\backslash\mathring{C})\right] + \left(\lambda(\mathring{C}') - \lambda(\mathring{C})\right)^2 \\ &\quad +2\left(\lambda(\mathring{C}') - \lambda(\mathring{C})\right)\times\left[\lambda(\mathring{C}\backslash\mathring{C}') - \lambda(\mathring{C}'\backslash\mathring{C})\right] \\ &= \lambda(\mathring{C}\Delta\mathring{C}') + \left(\lambda(\mathring{C}') - \lambda(\mathring{C})\right)^2 - 2\left(\lambda(\mathring{C}') - \lambda(\mathring{C})\right)^2 \\ &\le \lambda(\mathring{C}\Delta\mathring{C}'). \end{aligned}$$

Also we observe that

$$\begin{aligned} L|Z_i| &\leq 1+\left|\lambda(\mathring{C}') - \lambda(\mathring{C})\right| \\ &\leq 1+\lambda(\mathring{C}\Delta\mathring{C}'). \end{aligned}$$

If $\min\left\{\frac{u^2}{d_2^2(C,C')}, u\right\} = u$ then we have $u \leq \frac{u^2}{d_2^2(C,C')}$ i.e., $d_2^2(C,C') \leq u$ which implies that $n\lambda(\mathring{C}\Delta\mathring{C}') \leq u$. In that case

$$\begin{aligned} \sum_{i=1}^{n}\mathbb{E}[Z_i^2] + \frac{u}{3L}\left(1+\lambda(\mathring{C}\Delta\mathring{C}')\right) &\leq \frac{n}{L^2}\lambda(\mathring{C}\Delta\mathring{C}') + \frac{u}{3L}\left(1+\lambda(\mathring{C}\Delta\mathring{C}')\right) \\ &\leq \frac{u}{L^2} + \frac{2u}{3L} \\ &= \left(\frac{1}{L^2}+\frac{2}{3L}\right)u, \end{aligned} \tag{6.29}$$

where in the last inequality we used the facts that $n\lambda(\mathring{C}\Delta\mathring{C}') \leq u$ and $\lambda(\mathring{C}\Delta\mathring{C}') \leq 1$. On the other hand if $\min\left\{\frac{u^2}{d_2^2(C,C')}, u\right\} = \frac{u^2}{d_2^2(C,C')}$ then $n\lambda(\mathring{C}\Delta\mathring{C}') \geq u$, and we have,

$$\begin{aligned} \sum_{i=1}^{n}\mathbb{E}[Z_i^2] + \frac{u}{3L}\left(1+\lambda(\mathring{C}\Delta\mathring{C}')\right) &\leq \frac{n}{L^2}\lambda(\mathring{C}\Delta\mathring{C}') + \frac{2n}{3L}\lambda(\mathring{C}\Delta\mathring{C}') \\ &= \left(\frac{1}{L^2}+\frac{2}{3L}\right)n\lambda(\mathring{C}\Delta\mathring{C}') \\ &= \left(\frac{1}{L^2}+\frac{2}{3L}\right)d_2^2(C,C'). \end{aligned} \tag{6.30}$$

We can choose L suitably such that $2\left(\frac{1}{L^2}+\frac{2}{3L}\right) \leq 1$ and then using the Bernstein's inequality and (6.29), (6.30) we get

$$\begin{aligned} \mathbb{P}\left(|X_C - X_{C'}| > u\right) &\leq 2\exp\left\{-\frac{u^2/2}{\sum_{i=1}^{n}\mathbb{E}[Z_i^2] + Mu/3}\right\} \\ &\leq 2\exp\left\{-\min\left\{\frac{u^2}{d_2^2(C,C')}, u\right\}\right\} \end{aligned} \tag{6.31}$$

If $C = C'$ then obviously $\mathbb{P}\left(|X_C - X_{C'}| > u\right) = 0$ and hence using (6.31) we get that

$$\mathbb{P}\left(|X_C - X_{C'}| > u\right) \leq 2\exp\left\{-\min\left\{\frac{u^2}{d_2^2(C,C')}, \frac{u}{\delta(C,C')}\right\}\right\},$$

which is exactly the condition mentioned in the hypothesis of Theorem 6.4. Hence using Theorem 6.4 with $u_1 = \frac{1}{L}(\log n)^{\frac{3}{2}}$ and $u_2 = \frac{1}{\sqrt{L}}(\log n)^{\frac{3}{4}}$ we can say that with probability at least $1 - L\exp\left(-\frac{(\log n)^{\frac{3}{2}}}{L}\right)$ for each $C \in C(w,k)$ we have

$$\left|\sum_{i=1}^{n}\left(\mathbf{1}_{\mathring{C}}(X_i) - \mathbf{1}_{\mathring{C}'}(X_i)\right)\right| \leq L\left(\gamma_1(C(w,k),\delta) + \gamma_2(C(w,k),d_2)\right) + u_1D_1 + u_2D_2,$$

D_1 and D_2 are with respect to metrics δ and d_2 respectively.

To complete the proof we just need to show that the right hand side of the above equation is bounded by $L2^k\sqrt{n}(\log n)^{\frac{3}{4}}$, where L is just a universal constant and may not be same as that in the above equation. From the Lemma 6.17 we see that $\mathrm{card}(C(w,k)) \leq N_{k+l_1+1}$. So we can take an admissible sequence $\{\mathcal{A}_n\}$ of $C(w,k)$ such that $\mathcal{A}_n$ will be collection of singletons after $n = k + l_1 + 1$, thus

$$\begin{aligned}
\gamma_1(C(w,k),\delta) &\leq \sup_{C\in C(w,k)} \sum_{n\geq 0} 2^n \Delta(A_n(C),\delta) \\
&= \sum_{n\geq 0} 2^n \\
&\leq 4\times 2^{k+l_1} \\
&\leq 4\times 2^k \frac{\sqrt{n}}{(\log n)^{\frac{3}{4}}} \\
&\leq L2^k\sqrt{n}.
\end{aligned}$$

Also from (6.28) we have

$$\gamma_2(C(w,k),d_2) \leq L2^k\sqrt{n}(\log n)^{\frac{3}{4}}.$$

From the definition of D_is', $D_1 = 4\sum_{n\geq 0} e_n(C(w,k),\delta)$. Now $e_n(C(w,k),\delta) = 1$ for $n < k + l_1 + 1$ and $e_n(C(w,k),\delta) = 0$ for $n \geq k + l_1 + 1$. Therefore $D_1 \leq 4(k + l_1 + 1) \leq L\log n$. On the other hand since $d_2 = \sqrt{nd_1}$, using Proposition 6.20 we get that

$$\begin{aligned}
e_n(C(w,k),d_2) &= \sqrt{n}e_n(C(w,k),\sqrt{d_1}) \\
&\leq \sqrt{n}\left(e_n(C(w,k),d_1)\right)^{\frac{1}{2}} \\
&\leq L\sqrt{n}2^{-\frac{n}{2}}\gamma_1^{\frac{1}{2}}(C(w,k),d_1) \\
&\leq L\sqrt{n}2^k,
\end{aligned}$$

and thus $D_2 \leq L\sqrt{n}2^k$. Combining all these above results we get

$$\begin{aligned}
& L\left(\gamma_1(C(w,k),\delta) + \gamma_2(C(w,k),d_2)\right) + u_1D_1 + u_2D_2 \\
\leq\ & L'\left(2^k\sqrt{n} + 2^k\sqrt{n}(\log n)^{\frac{3}{4}}\right) + L(\log n)^{\frac{5}{2}} + L\sqrt{n}2^k(\log n)^{\frac{3}{4}} \\
\leq\ & L''\sqrt{n}2^k(\log n)^{\frac{3}{4}}.
\end{aligned}$$

This completes the proof of Lemma 5.4.

Related problems

In this thesis we considered only the two color matchings and matched edge lengths. There are several other kinds of matching problems namely one color matchings, stable matchings, etc.

One color matching: Here instead of having two independent point process we have only one point process and we want to form a graph out of it such that each vertex has degree exactly one i.e. we want to match each point to some other point. The problems are same as before i.e. we want to look at a typical matched edge length. The results for two color matchings in dimension more than two are true here for all dimensions [2].

Stable matching: In this case we may have one point process $\mathcal{R}$ (one color) (or two independent point processes $\mathcal{R}$ and $\mathcal{B}$ (two color)). Suppose we have a matching algorithm $\mathcal{M}$. We say that $\mathcal{M}$ is a stable matching algorithm if for any $x \in [\mathcal{R}]$ there doesn't exist $y \in [\mathcal{R}]$ other than $\mathcal{M}(x)$ (respectively, there doesn't exist $y \in [\mathcal{B}]$ other than $\mathcal{M}(x)$) such that $|x - y| < \min\{|x - \mathcal{M}(x)|, |y - \mathcal{M}(y)|\}$. The problems are same as before. Let X is a typical matched edge length. The following theorems are about behavior of X. [2]

Lower bounds [2] Any one color stable matching in dimensions $d \geq 1$ satisfies $\mathbb{E}X^d = \infty$, and any two color stable matching in dimensions $d \geq 1$ satisfies $\mathbb{E}X^d = \infty$.

Upper bounds [2] There exists one color stable matching in dimensions $d \geq 1$ such that $\mathbb{P}(X > r) \leq C(d)r^{-d}$, where $C(d)$ is a positive constant depends only on the dimension d. There exists two color stable matching in dimensions $d \geq 1$ such that $\mathbb{P}(X > r) \leq C(d)r^{-s(d)}$, where $C(d)$ and $s(d)$ are functions of d satisfying $C(d) \in (0, \infty)$; $s(d) \in (0, 1)$ and $s(1) = \frac{1}{2}$. For $d \geq 2$, the power $s(d)$ can be explicitly given by the unique solution in $(0, 1)$ of the equation

$$2\omega_d \int_1^2 (t-1)^{d-1} t^{-s}\, dt = \frac{\omega_{d-1}}{d-1} \int_0^2 \left(1 - \left(\frac{t}{2}\right)^2\right)^{\frac{d-1}{2}} t^{-s}\, dt,$$

where ω_d denotes the $(d-1)$ dimensional volume of the unit sphere in $\mathbb{R}^d$. It can be seen that $s(2) = 0.496$, $s(3) = 0.449 \ldots$(approximated up to three decimal places).

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