MAT 67 Max time 20mins **Quiz 5**

Name:____September 4, 2014

All unnecessary electronics must be turned off and out of sight. This means no cellular phones, iPods, wearing of headphones, or anything of the sort. This is a closed book and closed notes test.

Problem: Consider the following three vectors in $\mathbb{R}^3$

$$\begin{aligned} f_1 &= \frac{1}{\sqrt{2}}(1,0,-1) \\ f_2 &= (0,1,0) \\ f_3 &= \frac{1}{\sqrt{2}}(1,0,1). \end{aligned}$$

(i) *(8 points)* Prove that $\{f_1, f_2, f_3\}$ is a set of orthonormal vectors in $\mathbb{R}^3$ i.e., $\langle f_i, f_j \rangle = 0$ if $i \neq j$ and $\langle f_i, f_i \rangle = 1$ for all $i = 1, 2, 3$.

(ii) *(2 points)* Construct a 3×3 matrix A whose rows are given by f_1, f_2, and f_3.

(iii) *(2 points)* Compute the transpose of the matrix A.

(iv) *(6 points)* Find AA^T and A^TA *(You may use your results from part (i))*.

(v) *(2 points)* What is the inverse of A?

($\star$) *(Extra credit, 2 points)* What is the mathematical name of these type of matrices *(make a guess)*?

Solution:

(i) We compute

$$\begin{aligned} \langle f_1, f_2 \rangle &= \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 1 + \frac{-1}{\sqrt{2}} \cdot 0 = 0 \\ \langle f_1, f_3 \rangle &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 + \frac{-1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 0 \\ \langle f_2, f_3 \rangle &= 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0 + 0 \cdot \frac{1}{\sqrt{2}} = 0, \end{aligned}$$

and

$$\begin{aligned} \langle f_1, f_1 \rangle &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 + \frac{-1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} = \frac{1}{2} + \frac{1}{2} = 1 \\ \langle f_2, f_2 \rangle &= 0 + 1 + 0 = 1 \\ \langle f_3, f_3 \rangle &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Therefore $\{f_1, f_2, f_3\}$ is a set of orthonormal vectors.

(ii) The matrix A is given by

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(iii) Transpose of A is

$$A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(iv) Using the computations from *part (i)* we have

$$AA^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \langle f_1, f_1 \rangle & \langle f_1, f_2 \rangle & \langle f_1, f_3 \rangle \\ \langle f_2, f_1 \rangle & \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle \\ \langle f_3, f_1 \rangle & \langle f_3, f_2 \rangle & \langle f_3, f_3 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can also compute

$$A^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(v) From *part (iv)* we see that $AA^T = I = A^T A$. Therefore

$$A^{-1} = A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

($\star$) These type of matrices are called *orthogonal matrix*[1].

[1] A matrix M is called an orthogonal matrix if the rows of A are orthonormal set of vectors as well as the columns are also orthonormal set of vectors i.e., $MM^T = I = M^T M$