MAT 67 Max time: 1 hr 30 mins ~~**Final Exam**~~

Name:____September 11, 2014

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- check that you have all 6 pages of the exam. (The number of pages includes this cover page.)

During the exam:

- Keep your eyes on your own exam!
- No notes/books or electronics AT ALL!

Note that the exam length is exactly 1 hr 30 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	**Total**
Score					

Problem 1:*(20 points)* TRUE or FALSE? If the statement is true then give a proof. If the statement is false then give a counterexample.

(a)(5 points) Let A and B be 2×2 matrices then $AB = BA$.

Solution: FALSE. Let us consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}.$$

Clearly $AB \neq BA$.

(b)(5 points) $V = \{c(1,3) : \ c > 0\}$ is a vector space.

Solution: FALSE. It is not a vector space because the additive identity $(0,0,0) \notin V$ (Also notice that V does not have additive inverses).

(c)(5 points) If $\{v_1, v_2\} \subset \mathbb{R}^3$ is an orthonormal set of vectors in $\mathbb{R}^3$ then v_1, v_2 are independent.

Solution: TRUE. Let $c_1 v_1 + c_2 v_2 = 0$. Taking inner product with v_1 on both sides we have

$$\begin{aligned} & c_1\langle v_1, v_1\rangle + c_2\langle v_2, v_1\rangle = \langle 0, v_1\rangle \\ \Rightarrow \quad & c_1 = 0 \quad (\text{since } \langle v_1, v_1\rangle = 1, \langle v_2, v_1\rangle = 0). \end{aligned}$$

Similarly, taking inner product with v_2 we have $c_2 = 0$. Therefore $\{v_1, v_2\}$ are independent.

(d)(5 points) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ can not be injective.

Solution: TRUE. Using the dimension formula

$$\begin{aligned} & \dim(null(T)) + \dim(range(T)) = \dim(\mathbb{R}^3) \\ \Rightarrow \quad & \dim(null(T)) = 3 - \dim(range(T)). \end{aligned}$$

Since $range(T) \subset \mathbb{R}^2$, $\dim(range(T)) \leq 2$. Therefore $\dim(null(T)) \geq 1$ i.e., $null(T) \neq \{0\}$. So T is not injective.

Problem 2: *(20 points)* Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) *(2 points)* The matrix A can be considered as a linear map $A : \mathbb{R}^p \to \mathbb{R}^q$. What are the values of p and q?

(b) *(3 points)* Define the null space of the linear transformation A.

(c) *(8 points)* Solve the system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(d) *(3 points)* What is a basis of $null(A)$?

(e) *(2 points)* Then what is the dimension of the null space of A?

(f) *(2 points)* Therefore the dimension of $Range(A)$ is ______.

Solution:

(a) Since A is a 3×3 matrix, we have $p = 3, q = 3$.[1]

(b) $null(A) = \{v \in \mathbb{R}^3 : Av = 0\}$.

(c) From the matrix itself we can say that x_1, x_3 are pivot variables and x_2 is a free variable. Let us assign $x_2 = 1$. Then the system of equations become

$$\begin{aligned} x_1 + 2 + x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

Solution of the above system is $x_3 = 0, x_1 = -2$. We have already assigned $x_2 = 1$. Therefore $(-2, 1, 0)$ is a solution to the above system of equations. Note that there was only one free variable so we have only one independent solution, namely $(-2, 1, 0)$. All other solutions are given by $\{c(-2, 1, 0) : c \in \mathbb{R}\}$.

(d) From the above we can say that the basis of $null(A)$ is $\{(-2, 1, 0)\}$.

(e) Since there is only one vector in the basis of $null(A)$, $\dim(null(A)) = 1$.

(f) From the dimension formula

$$\dim(range(A)) = \dim(\mathbb{R}^3) - \dim(null(A)) = 3 - 1 = 2.$$

[1] If A is an $m \times n$ matrix then $p = n, q = m$

Problem 3:*(20 points)* Let U and W be subspaces of a finite dimensional vector space V over $\mathbb{R}$.

(a) *(3 points)* Take $u, w \in U \cap W$. Show that $au + bw \in U \cap W$ for any $a, b \in \mathbb{R}$.

(b) *(2 points)* Conclude that $U \cap W$ is a subspace of U as well as $U \cap W$ is also a subspace of W.

($\star$) Assume that $dim(U \cap W) = k$, $dim(U) = k + m$, and $dim(W) = k + n$.

(c) *(2 points)* Take a basis $\mathcal{B} = \{u_1, \ldots, u_k\}$ of $U \cap W$. Extend $\mathcal{B}$ to a basis $\mathcal{B}_U$ of U. Also extend $\mathcal{B}$ as a basis $\mathcal{B}_W$ of W.

(d) *(3 points)* Write down $B_U \cup \mathcal{B}_W$. How many vectors are there in $B_U \cup \mathcal{B}_W$?

(e) *(2 points)* Argue that $span(\mathcal{B}_U \cup \mathcal{B}_W) \subset U + W$.

(f) *(3 points)* Prove that $U + W \subset span(\mathcal{B}_U \cup \mathcal{B}_W)$.

(g) *(1 points)* Conclude that $span(\mathcal{B}_U \cup \mathcal{B}_W) = U + W$.

(h) *(4 points)* Conclude that $dim(U + W) \leq dim(U) + dim(W) - dim(U \cap W)$.

Solution:

(a) Let $u, w \in U \cap W$. Since $u, w \in U$ and it is given to us that U is a subspace, we have $au + bw \in U$ for any $a, b \in \mathbb{R}$. Also $u, w \in W$ and W is a subspace, therefore $au + bw \in W$ for any $a, b \in \mathbb{R}$. Consequently $au + bw \in U \cap W$.

(b) It is obvious that $U \cap W \subset U$ also $U \cap W \subset W$. From the part (a), we can conclude that $U \cap W$ is a subspace of U as well as $U \cap W$ is a subspace of W.

(c) Since $U \cap W$ is a subspace of U and $U \cap W$ is also a subspace of W we can extend the basis $\mathcal{B} = \{u_1, \ldots, u_k\}$ of $U \cap W$ to a basis $\mathcal{B}_U = \{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ of U. Also we can extend $\mathcal{B}$ to a basis $\mathcal{B}_W = \{u_1, \ldots, u_k, w_1, \ldots, w_n\}$ of W.

(d)

$$\mathcal{B}_U \cup \mathcal{B}_W = \{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}.$$

The number of vectors in $\mathcal{B}_U \cup \mathcal{B}_W$ is $m + n + k$.

(e) Since $u_1, \ldots, u_k, v_1, \ldots, v_m \in U$ and $w_1, \ldots, w_n \in W$, we have $u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n \in U + W$. Therefore $span\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\} \subset U + W$.

(f) Take a vector $v \in U + W$, we want to show that $v \in span(\mathcal{B}_U \cup \mathcal{B}_W)$. Since $v \in U + W$, by the definition of $U + W$ we can write $v = u + w$, where $u \in U$ and $w \in W$. Since $u \in U$ and $\mathcal{B}_U$ is a basis of U, there exist scalars $a_1, \ldots, a_k, b_1, \ldots, b_m \in \mathbb{R}$ such that

$$u = a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_m v_m.$$

Similarly we know that $w \in W$ and $\mathcal{B}_W$ is a basis of W. Therefore there exist scalars $c_1, \ldots, c_k, d_1, \ldots, d_n \in \mathbb{R}$ such that

$$w = c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_n w_n.$$

So finally we have

$$\begin{aligned} v &= u + w \\ &= [a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_m v_m] \\ &\quad +[c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_n w_n] \\ &= (a_1 + c_1) u_1 + \cdots + (a_k + c_k) u_k + b_1 v_1 + \cdots + b_m v_m + d_1 w_1 + \cdots + d_n w_n. \end{aligned}$$

Therefore $v \in span(\mathcal{B}_U \cup \mathcal{B}_W)$. Consequently $U + W \subset span(B_U \cup B_W)$.

(g) From *part (e)* and *part (f)* we can conclude that $span(\mathcal{B}_U \cup \mathcal{B}_W) = U + W$.

(h) Since $U+W = span(\mathcal{B}_U \cup \mathcal{B}_W)$ and $\mathcal{B}_U \cup \mathcal{B}_W$ has $k+m+n$ many vectors, we have $\dim(U+W) \leq k+m+n$.[2] On the other hand from the assumption $(\star)$ we see that

$$\dim(U) + \dim(W) - \dim(U \cap W) = (k+m) + (k+n) - k = k + m + n.$$

Therefore

$$\dim(U+W) \leq \dim(U) + \dim(W) - \dim(U \cap W).$$

[2] If the vectors in $\mathcal{B}_U \cup \mathcal{B}_W$ are independent then we would have $\dim(U+W) = k+m+n$. But we do not know whether the vectors in $\mathcal{B}_U \cup \mathcal{B}_W$ are independent or not. If the vectors in $\mathcal{B}_U \cup \mathcal{B}_W$ are dependent then we can throw out the dependent vectors from $\mathcal{B}_U \cup \mathcal{B}_W$ and still the remaining vectors will span $U+W$. The dimension of $U+W$ is equal to the number of *independent* vectors in $\mathcal{B}_U \cup \mathcal{B}_W$ which is less or equal to $k+m+n$. So in any case $\dim(U+W) \leq k+m+n$.

Problem 4: *(20 points)* Consider the vectors $f_1 = (1,0,0), f_2 = (1,2,0), f_3 = (1,1,3)$.

(a) *(2 points)* Construct a 3×3 matrix M such that columns of A are given by the above vectors.

(b) *(6 points)* Is M a normal matrix?

(c) *(2 points)* Compute the determinant of M.

(d) *(2 points)* Is $\{f_1, f_2, f_3\}$ a basis of $\mathbb{R}^3$.

(e) *(8 points)* Using Gram-Schmidt Orthogonalization on $\{f_1, f_2, f_3\}$, find an othonormal basis of $\mathbb{R}^3$.

Solution:

(a) The matrix A is given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) Since all the entries of A are real numbers,

$$A^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}.$$

We can see that

$$\begin{aligned} AA^* &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 5 & 3 \\ 3 & 3 & 9 \end{bmatrix} \\ A^*A &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 3 \\ 1 & 2 & 11 \end{bmatrix}. \end{aligned}$$

Since $AA^* \neq A^*A$, the matrix A is not normal.

(c)

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

(d) Since $\det(A) \neq 0$, the columns of A i.e., f_1, f_2, f_3 are independent. So $\{f_1, f_2, f_3\}$ is a set of three independent vectors in a three dimensional vector space $\mathbb{R}^3$. Therefore $\{f_1, f_2, f_3\}$ is a basis of $\mathbb{R}^3$.

(e)

$$\begin{aligned} \tilde{f}_1 &= (1,0,0). \\ \tilde{f}_2 &= f_2 - \frac{\langle f_2, \tilde{f}_1 \rangle}{\langle \tilde{f}_1, \tilde{f}_1 \rangle} \tilde{f}_1 \\ &= (1,2,0) - \frac{1}{1}(1,0,0) \\ &= (0,2,0). \\ \tilde{f}_3 &= f_3 - \frac{\langle f_3, \tilde{f}_2 \rangle}{\langle \tilde{f}_2, \tilde{f}_2 \rangle} \tilde{f}_2 - \frac{\langle f_3, \tilde{f}_1 \rangle}{\langle \tilde{f}_1, \tilde{f}_1 \rangle} \tilde{f}_1 \\ &= (1,1,3) - \frac{2}{4}(0,2,0) - \frac{1}{1}(1,0,0) \\ &= (0,0,3). \end{aligned}$$

Now the vectors $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ are orthogonal to each other but they are not normalized. To normalize them we divide each vector by its norm. And finally the orthonormal basis we obtained by Gram-Schmidt orthogonalization is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.