MAT 67 Max time: 1 hr 30 mins ~~**Final Exam**~~

Name:____ September 11, 2014

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- check that you have all 6 pages of the exam. (The number of pages includes this cover page.)

During the exam:

- Keep your eyes on your own exam!
- No notes/books or electronics AT ALL!

Note that the exam length is exactly 1 hr 30 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	**Total**
Score					

Problem 1: *(20 points)* TRUE or FALSE? If the statement is true then give a proof. If the statement is false then give a counterexample.

(a)(5 points) Let A and B be 2×2 matrices then $AB = BA$.

Solution:

(b)(5 points) $V = \{c(1,3) : \ c > 0\}$ is a vector space.

Solution:

(c)(5 points) If $\{v_1, v_2\} \subset \mathbb{R}^3$ is an orthonormal set of vectors in $\mathbb{R}^3$ then v_1, v_2 are independent.

Solution:

(d)(5 points) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ can not be injective.

Solution:

Problem 2: *(20 points)* Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) *(2 points)* The matrix A can be considered as a linear map $A : \mathbb{R}^p \to \mathbb{R}^q$. What are the values of p and q?

(b) *(3 points)* Define the null space of the linear transformation A.

(c) *(8 points)* Solve the system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(d) *(3 points)* What is a basis of $null(A)$?

(e) *(2 points)* Then what is the dimension of the null space of A?

(f) *(2 points)* Therefore the dimension of $Range(A)$ is ______.

Solution:

Problem 3: *(20 points)* Let U and W be subspaces of a finite dimensional vector space V over $\mathbb{R}$.

(a) *(3 points)* Take $u, w \in U \cap W$. Show that $au + bw \in U \cap W$ for any $a, b \in \mathbb{R}$.

(b) *(2 points)* Conclude that $U \cap W$ is a subspace of U as well as $U \cap W$ is also a subspace of W.

($\star$) Assume that $dim(U \cap W) = k$, $dim(U) = k + m$, and $dim(W) = k + n$.

(c) *(2 points)* Take a basis $\mathcal{B} = \{u_1, \ldots, u_k\}$ of $U \cap W$. Extend $\mathcal{B}$ to a basis $\mathcal{B}_U$ of U. Also extend $\mathcal{B}$ as a basis $\mathcal{B}_W$ of W.

(d) *(3 points)* Write down $B_U \cup \mathcal{B}_W$. How many vectors are there in $B_U \cup \mathcal{B}_W$?

(e) *(2 points)* Argue that $span(\mathcal{B}_U \cup \mathcal{B}_W) \subset U + W$.

(f) *(3 points)* Prove that $U + W \subset span(\mathcal{B}_U \cup \mathcal{B}_W)$.

(g) *(1 points)* Conclude that $span(\mathcal{B}_U \cup \mathcal{B}_W) = U + W$.

(h) *(4 points)* Conclude that $dim(U + W) \leq dim(U) + dim(W) - dim(U \cap W)$.

Solution:

Problem 4: *(20 points)* Consider the vectors $f_1 = (1, 0, 0), f_2 = (1, 2, 0), f_3 = (1, 1, 3)$.

(a) *(2 points)* Construct a 3×3 matrix M such that columns of A are given by the above vectors.

(b) *(6 points)* Is M a normal matrix?

(c) *(2 points)* Compute the determinant of M.

(d) *(2 points)* Is $\{f_1, f_2, f_3\}$ a basis of $\mathbb{R}^3$.

(e) *(8 points)* Using Gram-Schmidt Orthogonalization on $\{f_1, f_2, f_3\}$, find an othonormal basis of $\mathbb{R}^3$.

Solution: