MAT 67 **Homework 5**
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Problem 9.2: Let us consider two vectors $u, v \in \mathbb{R}^n$, where

$$\begin{aligned} u &= \left(a_1, \sqrt{2}a_2, \sqrt{3}a_3, \ldots, \sqrt{n}a_n\right) \ \textit{i.e.,}\ u_k = \sqrt{k}a_k \ \forall\ k = 1, \ldots, n \\ v &= \left(b_1, \frac{b_2}{\sqrt{2}}, \frac{b_3}{\sqrt{3}}, \ldots, \frac{b_n}{\sqrt{n}}\right) \ \textit{i.e.,}\ v_k = \frac{b_k}{\sqrt{k}} \ \forall\ k = 1, \ldots, n. \end{aligned}$$

Applying the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\langle u, v\rangle| &\leq \|u\|\|v\| \\ \Rightarrow |\langle u, v\rangle|^2 &\leq \|u\|^2\|v\|^2 \\ \Rightarrow \left(\sum_{k=1}^{n} u_k v_k\right)^2 &\leq \left(\sum_{k=1}^{n} u_k^2\right)\left(\sum_{k=1}^{n} v_k^2\right) \\ \Rightarrow \left(\sum_{k=1}^{n} a_k b_k\right)^2 &\leq \left(\sum_{k=1}^{n} k a_k^2\right)\left(\sum_{k=1}^{n} \frac{b_k^2}{k}\right). \end{aligned}$$

Problem 9.4: Since V is an inner product space over $\mathbb{R}$, we have $\langle v, u\rangle = \langle u, v\rangle$. Therefore

$$\begin{aligned} \|u+v\|^2 - \|u-v\|^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\ &= (\langle u, u+v\rangle + \langle v, u+v\rangle) - (\langle u, u-v\rangle - \langle v, u-v\rangle) \\ &= (\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle) - (\langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle) \\ &= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle - \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle \\ &= \langle u, v\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, u\rangle \quad (1) \\ &= 4\langle u, v\rangle \quad (\text{since } \langle v, u\rangle = \langle u, v\rangle). \quad (2) \end{aligned}$$

So we have the result

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

Problem 9.5: In this problem, V is an inner product space over $\mathbb{C}$. Therefore we have $\langle v, u\rangle = \overline{\langle u, v\rangle}$. Proceeding as *Problem 9.4* and using the computation up to equation (1) we have[1]

$$\begin{aligned}
\|u+v\|^2 - \|u-v\|^2 &= \langle u,v\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,u\rangle \\
&= 2(\langle u,v\rangle + \langle v,u\rangle) \\
&= 2(\langle u,v\rangle + \overline{\langle u,v\rangle}) \quad (\text{since } \langle v,u\rangle = \overline{\langle u,v\rangle}) \\
&= 4Re\langle u,v\rangle \quad (\text{since } z+\bar{z} = 2Re(z), \text{ for any complex number } z) \qquad (3)
\end{aligned}$$

On the other hand, we know that $\langle cu, v\rangle = c\langle u,v\rangle$ and $\langle u, dv\rangle = \bar{d}\langle u,v\rangle$. Using these two properties we have

$$\begin{aligned}
\|u+iv\|^2 - \|u-iv\|^2 &= \langle u+iv, u+iv\rangle - \langle u-iv, u-iv\rangle \\
&= (\langle u, u+iv\rangle + i\langle v, u+iv\rangle) - (\langle u, u-iv\rangle - i\langle v, u-iv\rangle) \\
&= (\langle u,u\rangle + \bar{i}\langle u,v\rangle + i\langle v,u\rangle + i\bar{i}\langle v,v\rangle) - (\langle u,u\rangle - \bar{i}\langle u,v\rangle - i\langle v,u\rangle + i\bar{i}\langle v,v\rangle) \\
&= \langle u,u\rangle - i\langle u,v\rangle + i\langle v,u\rangle + \langle v,v\rangle - \langle u,u\rangle - i\langle u,v\rangle + i\langle v,u\rangle - \langle v,v\rangle \\
&= -i\langle u,v\rangle + i\langle v,u\rangle - i\langle u,v\rangle + i\langle v,u\rangle \\
&= -2i(\langle u,v\rangle - \langle v,u\rangle) \\
&= -2i(\langle u,v\rangle - \overline{\langle u,v\rangle}) \\
&= -2i\cdot(2iIm\langle u,v\rangle) \quad (\text{since } z-\bar{z} = 2iIm(z), \text{ for any complex number } z) \\
&= 4Im\langle u,v\rangle \qquad (4)
\end{aligned}$$

Hence using the equations (3) and (4) we have

$$\frac{\|u+v\|^2 - \|u-v\|^2}{4} + i\frac{\|u+iv\|^2 - \|u-iv\|^2}{4} = Re\langle u,v\rangle + iIm\langle u,v\rangle = \langle u,v\rangle.$$

Problem 9.6: Let $\{u_1,\ldots,u_k\}$ be an orthonormal basis of U, and $\{w_1,\ldots,w_m\}$ be an orthonormal basis of $U^\perp$. We would like to show that $\{u_1,\ldots,u_k,w_1,\ldots,w_m\}$ is a basis of V.

First of all, since $\{u_1,\ldots,u_k,w_1,\ldots,w_m\} \subset V$, we have

$$span\{u_1,\ldots,u_k,w_1,\ldots,w_m\} \subset V. \qquad (5)$$

Conversely, we will show that $V \subset span\{u_1,\ldots,u_k,w_1,\ldots,w_m\}$. Let us take $v \in V$. From theorem 9.6.2, we know that $V = U \oplus U^\perp$. Therefore we can uniquely decompose v as $v = u + w$, where $u \in U$ and $w \in W$. Since $\{u_1,\ldots,u_k\}, \{w_1,\ldots,w_m\}$ are bases of U and $U^\perp$ respectively, there exist scalars $a_1,\ldots,a_k,b_1,\ldots,b_m \in \mathbb{F}$ such that

$$\begin{aligned}
u &= a_1u_1 + \cdots + a_ku_k \\
w &= b_1w_1 + \cdots + b_mw_m.
\end{aligned}$$

Therefore

$$v = a_1u_1 + \cdots + a_ku_k + b_1w_1 + \cdots + b_mw_m \in span\{u_1,\ldots,u_k,w_1,\ldots,w_m\}.$$

[1] Note that we can not use the equation (2). Because we are working on complex vector space. So we don't have $\langle v,u\rangle = \langle u,v\rangle$ instead we have $\langle v,u\rangle = \overline{\langle u,v\rangle}$.

So we have

$$V \subset span\{u_1, \dots, u_k, w_1, \dots, w_m\} \tag{6}$$

Using (5) and (6) we have

$$V = span\{u_1, \dots, u_k, w_1, \dots, w_m\}.$$

To complete the proof, we have to show that $\{u_1, \dots, u_k, w_1, \dots, w_m\}$ is a set of independent vectors. Let

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_m w_m = 0.$$

Taking inner product of both sides of the above equation with u_i we have $c_i \langle u_i, u_i \rangle = 0$ i.e, $c_i = 0$.[2] Similarly taking inner product with w_j we have $d_j = 0$. These are true for any $i = 1, \dots, k$ and $j = 1, \dots, m$. Therefore $\{u_1, \dots, u_k, w_1, \dots, w_m\}$ is a set of independent vectors. So $\{u_1, \dots, u_k, w_1, \dots, w_m\}$ is a basis of V. Consequently,

$$dim(V) = k + m = dim(U) + dim(U^\perp).$$

Alternative method

Using Theorem 9.6.2 (1), we know that $V = U + U^\perp$. Therefore using Theorem 5.4.6 we have

$$dim(V) = dim(U + U^\perp) = dim(U) + dim(U^\perp) - dim(U \cap U^\perp).$$

But Theorem 9.6.2 (2) says that $U \cap U^\perp = \{0\}$. Therefore $dim(U \cap U^\perp) = 0$. Consequently,

$$dim(V) = dim(U) + dim(U^\perp).$$

Problem 9.7: Suppose $U = V$. We will show that $U^\perp = \{0\}$. Let $w \in U^\perp$, then by definition $\langle w, u \rangle = 0$ for all $u \in U$. But we have assumed that $U = V$. Therefore we have $\langle w, v \rangle = 0$ for all $v \in V$. In particular $\langle w, w \rangle = 0$ (because $w \in U^\perp \subset V$ i.e., $w \in V$). Which implies that $\|w\|^2 = 0$. Therefore $w = 0$ i.e., $U^\perp = \{0\}$.

Conversely, suppose $U^\perp = \{0\}$. We will show that $U = V$. Using the theorem 9.6.2 we have

$$V = U \oplus U^\perp = U \oplus \{0\} = U.$$

[2] We are using the facts that $\{u_1, \dots, u_k\}$ is an orthogonal basis, therefore $\langle u_i, u_j \rangle = 0$ if $i \neq j$ and $\langle u_i, u_i \rangle = 1$. Also for $u_i \in U$ and $w_j \in U^\perp$, we have $\langle u_i, w_j \rangle = 0$

Alternative method

Using problem 9.6 we have

$$dim(V) = dim(U) + dim(U^{\perp}).$$

If $U = V$ then $dim(U) = dim(V)$. Therefore $dim(U^{\perp}) = 0$. Which implies that $U^{\perp} = \{0\}$.

Conversely, if $U^{\perp} = \{0\}$, then $dim(U^{\perp}) = 0$. Therefore $dim(U) = dim(V)$. From homework problem 5.4 we can conclude that $U = V$.