MAT 67 | **Homework 4**
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Problem 7.8: The claim is false. Consider the vector space $V = \mathbb{R}^2$, and consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(x, y) = (y, x)$. The basis of T with respect to the canonical basis is $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The matrix T has all zeros in the diagonal. However $det(T) = -1 \neq 0$. Therefore T is invertible.

Problem 7.9: The claim is false. Consider the matrix $T = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. All diagonal elements of T are nonzero. But $det(T) = 0$, therefore T is not invertible.

Problem 7.10: Let $dim(V) = n$. According to the given condition T has n distinct eigenvalues. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalues of T, and $v_1, v_2, \ldots, v_n$ be the corresponding eigenvectors i.e., $T(v_i) = \lambda_i v_i$ for all $i = 1, 2, \ldots, n$. Since the eigenvalues are distinct, $\{v_1, v_2, \ldots, v_n\}$ is a set of n linearly independent vectors in V. But $dim(V) = n$, therefore $\{v_1, v_2, \ldots, v_n\}$ forms a basis of V.

It is also given that all eigenvectors of T are also the eigenvectors of S (possibly different eigenvalues). Therefore there exist scalars $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{F}$ such that $S(v_i) = \mu_i v_i$ for all $i = 1, 2, \ldots, n$.

We see that for all $i = 1, 2, \ldots, n$

$$\begin{aligned} & T \circ S(v_i) = T(S(v_i)) = T(\mu_i v_i) = \mu_i T(v_i) = \mu_i \lambda_i v_i. \\ \text{and} \quad & S \circ T(v_i) = S(T(v_i) = S(\lambda_i v_i) = \lambda_i S(v_i) = \lambda_i \mu_i v_i. \end{aligned}$$

Therefore $T \circ S(v_i) = S \circ T(v_i)$ for all $i = 1, 2, \ldots, n$.

Let $v \in V$ be an arbitrarily chosen vector. Since $\{v_1, v_2, \ldots, v_n\}$ is a basis of V, there exist scalars $c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$. We have just shown that $T \circ S(v_i) = S \circ T(v_i)$ for all $i = 1, 2, \ldots, n$. Therefore

$$\begin{aligned} T \circ S(v) &= T \circ S(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\ &= c_1 T \circ S(v_1) + c_2 T \circ S(v_2) + \cdots + c_n T \circ S(v_n) \\ &= c_1 S \circ T(v_1) + c_2 S \circ T(v_2) + \cdots + c_n S \circ T(v_n) \\ &= S \circ T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\ &= S \circ T(v). \end{aligned}$$

Therefore $T \circ S(v) = S \circ T(v)$ for any $v \in V$. In other words $T \circ S = S \circ T$.

Problem 8.2: We know that any matrix B is invertible if and only if $det(B) \neq 0$. We also know that $det(B^T) = det(B)$ and $det(BC) = det(B)det(C)$. Using these two properties we can say that

$$\begin{aligned} & A^T A \text{ is invertible} \\ \iff & \det(A^T A) \neq 0 \\ \iff & \det(A^T)\det(A) \neq 0 \\ \iff & [\det(A)]^2 \neq 0 \\ \iff & \det(A) \neq 0 \\ \iff & A \text{ is invertible.} \end{aligned}$$

Problem 8.3: The statement is false. For example, consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $det(A) = 1$, $det(B) = 0$. But

$$A + B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and $det(A + B) = 3 \neq det(A) + det(B)$.

Problem 8.4: The statement is false. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $r = 3$. Then $det(A) = 1$ and $det(3A) = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9$. But $3det(A) = 3$. Therefore $det(3A) \neq 3det(A)$.