MAT 67 **Homework 2**
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Problem 4.2: Let us take two arbitrary vectors $v_1, v_2 \in W_1 \cap W_2$ and two scalars $a, b \in \mathbb{F}$. We need to show that $av_1 + bv_2 \in W_1 \cap W_2$.

Since $v_1, v_2 \in W_1 \cap W_2$, we have $v_1 \in W_1, v_1 \in W_2, v_2 \in W_1, v_2 \in W_2$. Now since W_1 is a vector space and $v_1, v_2 \in W_1$ we must have $av_1 + bv_2 \in W_1$. For the similar reason, we also have $av_1 + bv_2 \in W_2$. Consequently, $av_1 + bv_2 \in W_1 \cap W_2$.

Problem 4.4: The claim is false.

Consider the vector space $\mathbb{R}^2$ over the field $\mathbb{R}$. Now define

$$W_1 := span\{(1,0)\} = \{a(1,0) : a \in \mathbb{R}\} \tag{1}$$
$$W_2 := span\{(0,1)\} = \{b(0,1) : b \in \mathbb{R}\} \tag{2}$$
$$W_3 := span\{(1,1)\} = \{c(1,1) : c \in \mathbb{R}\}. \tag{3}$$

Since $(1,0)$ and $(1,1)$ are two independent vectors in $\mathbb{R}^2$, $span\{(1,0),(1,1)\} = \mathbb{R}^2$. But $span\{(1,0),(1,1)\} = \{c_1(1,0) + c_2(0,1) : c_1, c_2 \in \mathbb{R}\} = W_1 + W_3$. Therefore $\mathbb{R}^2 = W_1 + W_3$. Also we can see that $W_1 \cap W_3 = \{(0,0)\}$

[Take $(x_1,x_2) \in W_1 \cap W_3$. Then $(x_1,x_2) \in W_1$ as well as $(x_1,x_2) \in W_3$. Therefore there exists $a, c \in \mathbb{R}$ such that $(x_1,x_2) = a(1,0)$ and $(x_1,x_2) = c(1,1)$. Which implies that $a(1,0) = c(1,1)$ i.e, $a = 0, c = 0$. Therefore $(x_1,x_2) = (0,0)$. So any vector $(x_1,x_2) \in W_1 \cap W_3$ is equal to $(0,0)$. Therefore $W_1 \cap W_3 = \{(0,0)\}$].

So using the the proposition 4.4.7 we can say that $\mathbb{R}^2 = W_1 \oplus W_3$.

Similarly, $\mathbb{R}^2 = W_2 \oplus W_3$. Therefore $W_1 \oplus W_3 = \mathbb{R}^2 = W_2 \oplus W_3$. But obviously from definitions (1), (3) we can see that $W_1 \neq W_3$.

Problem 5.1: Let us define

$$\begin{aligned} w_1 &:= v_1 - v_2 \\ w_2 &:= v_2 - v_3 \\ &\vdots \\ w_{n-1} &:= v_{n-1} - v_n \\ w_n &:= v_n. \end{aligned}$$

We want to show that $span\{w_1, w_2, \ldots, w_n\} = span\{v_1, v_2, \ldots, v_n\} = V$. Since each w_i is a linear combination of the vectors $v_1, v_2, \ldots, v_n$, we have $w_i \in span\{v_1, v_2, \ldots, v_n\}$ for all $i = 1, 2, \ldots, n$. Therefore by Lemma 5.1.2 we have

$$span\{w_1, w_2, \ldots, w_n\} \subset span\{v_1, v_2, \ldots, v_n\}.$$

Now we want to show that $span\{v_1, v_2, \dots, v_n\} \subset span\{w_1, w_2, \dots, w_n\}$. Let us take a vector $v \in span\{v_1, v_2, \dots, v_n\}$. Then v can be written as a linear combination of $v_1, v_2, \dots, v_n$ i.e., there exist scalars $c_1, c_2, \dots, c_n \in \mathbb{F}$ such that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

Our goal is to write v as a linear combination of $w_1, w_2, \dots, w_n$. So first let us write each v_i as a linear combination of $w_1, w_2, \dots, w_n$. We notice that

$$\begin{aligned} v_n &= w_n \\ v_{n-1} &= w_{n-1} + w_n \\ &\vdots \\ v_2 &= w_2 + w_3 + \cdots + w_n \\ v_1 &= w_1 + w_2 + \cdots + w_n. \end{aligned}$$

Therefore we have

$$\begin{aligned} v &= c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \\ &= c_1(w_1 + w_2 + \cdots + w_n) + c_2(w_2 + w_3 + \cdots + w_n) + \cdots + c_n w_n \\ &= c_1 w_1 + (c_1 + c_2) w_2 + (c_1 + c_2 + c_3) w_3 + \cdots + (c_1 + c_2 + \cdots + c_n) w_n. \end{aligned}$$

The last line is a linear combination of $w_1, w_2, \dots, w_n$. Therefore $v \in span\{w_1, w_2, \dots, w_n\}$, and hence $span\{v_1, v_2, \dots, w_n\} \subset span\{w_1, w_2, \dots, w_n\}$.

Problem 5.3: *[This problem relies on the same idea as problem 4.4]* Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V. Define the spaces

$$\begin{aligned} U_1 &= span\{v_1\} \\ U_2 &= span\{v_2\} \\ &\vdots \\ U_n &= span\{v_n\}. \end{aligned}$$

Since each $v_i \in V$, each U_i is a subspace of V. Now since $\{v_1, v_2, \dots, v_n\}$ is a basis of V, each $v \in V$ can be written as a *unique* linear combination of $v_1, v_2, \dots, v_n$ i.e., $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$. But $c_i v_i \in U_i$ for each $i = 1, 2, \dots, n$. Therefore $V = U_1 \oplus +U_2 \oplus + \cdots + \oplus U_n$.

Problem 5.4: Since it is given that U is a subspace of V, we know that $U \subset V$. So if we want to show that $U = V$, we need to show that $V \subset U$.

Now let $dim(V) = n$, then we have $dim(U) = n$ (since it is given that $dim(U) = dim(V)$). Now let $\{u_1, u_2, \dots, u_n\}$ be a basis of U. Then $\{u_1, u_2, \dots, u_n\}$ is a set of independent of vectors in U. But it is given that $U \subset V$. Therefore $\{u_1, u_2, \dots, u_n\}$ is also a set of

n independent vectors in V. But $dim(V) = n$, therefore $\{u_1, u_2, \dots, u_n\}$ is also a basis of V (see Theorem 5.4.4). Which implies that any vector $v \in V$ can be written as $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$ uniquely for some scalars $c_i \in \mathbb{F}$. But $\{u_1, u_2, \dots, u_n\}$ is also a basis of U, therefore $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n \in U$. Consequently, $V \subset U$.

Problem 5.6: Let $\{u_1, u_2, u_3, u_4, u_5\}$ and $\{v_1, v_2, v_3, v_4, v_5\}$ be bases of U and V respectively. Now suppose if possible $U \cap V = \{0\}$. We will find a contradiction. Consider the following equation

$$c_1u_1 + \cdots + c_5u_5 + d_1v_1 + \cdots + d_5v_5 = 0. \tag{4}$$

The above equation can be written as

$$c_1u_1 + \cdots + c_5u_5 = (-d_1)v_1 + \cdots + (-d_5)v_5. \tag{5}$$

From the above equation we notice that the vector $c_1u_1 + \cdots + c_5u_5$ is a linear combination of the vectors $v_1, \dots, v_5$, therefore $c_1u_1 + \cdots + c_5u_5 \in V$. But naturally $c_1u_1 + \cdots + c_5u_5 \in U$. Therefore $c_1u_1 + \cdots + c_5u_5 \in U \cap V$. But we assumed that $U \cap V = \{0\}$. Therefore

$$c_1u_1 + \cdots + c_5u_5 = 0. \tag{6}$$

Since $\{u_1, u_2, u_3, u_4, u_5\}$ is a basis of U, $u_1, \dots, u_5$ are linearly independent. Therefore (6) is true only of $c_1 = 0, \dots, c_5 = 0$. Then (5) gives us

$$(-d_1)v_1 + \cdots + (-d_5)v_5 = 0,$$

which is true only if $d_1 = 0, \dots, d_5 = 0$ (since $\{v_1, \dots, v_5\}$ is a basis i.e., they are independent). So finally we have $c_1 = 0, \dots, c_5 = 0, d_1 = 0, \dots, d_5 = 0$. Therefore from (4) we can say that $u_1, \dots, u_5, v_1, \dots, v_5$ are linearly independent. But it is a contradiction, because all of them belong to $\mathbb{R}^9$ and we can not have ten linearly independent vectors in $\mathbb{R}^9$. Therefore we must have $U \cap V \neq \{0\}$.

<u>Alternative Proof</u>

Since both U and V are subspaces of $\mathbb{R}^9$, $U + V$ is also a subspace of $\mathbb{R}^9$. Now using the Theorem 5.4.6 we have

$$\begin{aligned} dim(U+V) &= dim(U) + dim(V) - dim(U \cap V) \\ &= 5 + 5 - dim(U \cap V) \\ i.e., \ \dim(U \cap V) &= 10 - dim(U+V). \end{aligned}$$

Since $U+V$ is a subspace of $\mathbb{R}^9$, $dim(U+V) \leq 9$. Therefore $dim(U \cap V) = 10 - dim(U+V) \geq 10 - 9 = 1$. Since $dim(U \cap V) \geq 1$, we must have $U \cap V \neq \{0\}$.

Remark: The geometric idea: Let U be an m dimensional subspace of an n dimensional vector space W. Now if we want to roam inside W but don't want to hit U then we have only $n-m$ degrees of freedom. In the above problem $W = \mathbb{R}^9$ and U is a five dimensional subspace. Now if we want to roam inside V which avoids U (means $V \cap U = \{0\}$) then V can be at most $9-5=4$ dimensional subspace. If V is five dimensional then it must intersect U.

Problem 5.7: *[This problem uses the same idea as problem 5.6]* We will prove it by induction. $*\backslash$ Let U and W be two subspaces of V. We will show that $dim(U+W) \leq dim(U)+dim(W)$. Let $dim(U)=k$, $dim(W)=l$ and $\{u_1,\dots,u_k\}$, $\{w_1,\dots,w_l\}$ be bases of U and W respectively. Then any vector $u \in U$ is a linear combination of $u_1,\dots,u_k$ and any vector $w \in W$ is a linear combination of $w_1,\dots,w_l$. Therefore any vector $u+w \in U+W$ is a linear combination of $u_1,\dots,u_k,w_1,\dots,w_l$. Therefore $span\{u_1,\dots,u_k,w_1,\dots,w_l\} = U+W$.

Now $u_1,\dots,u_k,w_1,\dots,w_l$ may not be linearly independent. But we can throw out the dependent vectors from $\{u_1,\dots,u_k,w_1,\dots,w_l\}$ and make a basis of $U+W$ out of it. Therefore basis of $U+W$ contains at most $k+l$ vectors. Consequently,

$$dim(U+W) \leq k+l = dim(U)+dim(W) \quad *\backslash \tag{7}$$

Using the above result we can conclude that

$$dim(U_1+U_2) \leq dim(U_1)+dim(U_2).$$

Now suppose the statement is true for $m-1$ subspaces i.e.,

$$dim(U_1+\cdots+U_{m-1}) \leq dim(U_1)+\cdots+dim(U_{m-1}). \tag{8}$$

We want to show that the statement is true for m subspaces. Using the result (7) and the above equation (8) we can conclude that

$$\begin{aligned} dim(\underbrace{U_1+\cdots+U_{m-1}}_{U}+\underbrace{U_m}_{W}) &\leq dim(U_1+\cdots+U_{m-1})+dim(U_m) \\ &\leq dim(U_1)+\cdots+dim(U_{m-1})+dim(U_m) \quad \text{(using (8))}. \end{aligned}$$

Remark: The result in (7) can also be proved by using Theorem 5.4.6. Namely, we can say that

$$dim(U+W) = dim(U)+dim(W)-dim(U\cap W) \leq dim(U)+dim(W).$$

However, the text marked inside $*\backslash \dots *\backslash$ explains the main idea behind Theorem 5.4.6.

Problem 6.5: Let $dim(V)=n$ and $dim(null(T))=m$. Let $\{v_1,\dots,v_m\}$ be a basis of $null(T)$. Since $null(T)$ is a subspace of V, by the basis extension theorem we can extend this set $\{v_1,\dots,v_m\}$ to a basis of V. Let $\{v_1,\dots,v_m,u_1,\dots,u_{n-m}\}$ be a basis of V.

Now define

$$U = span\{u_1, \dots, u_{n-m}\}.$$

We will prove that

$$U \cap null(T) = \{0\} \text{ and } range(T) = \{T(u) | u \in U\}.$$

"First of all, we know that $\{v_1, \dots, v_m, u_1, \dots, u_{n-m}\}$ is a set of independent vectors (because it is a basis of V). Now if there is some vector $u \in U \cap null(T)$, then u can be written as

$$\begin{aligned} u &= c_1 u_1 + \cdots + c_{n-m} u_{n-m} \\ \text{also } u &= d_1 v_1 + \cdots + d_m v_m. \end{aligned}$$

(Because $\{u_1, \dots, u_{n-m}\}$ is a basis of U and $\{v_1, \dots, v_m\}$ is a basis of $null(T)$). Therefore we have

$$\begin{aligned} & c_1 u_1 + \cdots + c_{n-m} u_{n-m} = d_1 v_1 + \cdots + d_m v_m \\ i.e., \quad & c_1 u_1 + \cdots + c_{n-m} u_{n-m} + (-d_1) v_1 + \cdots + (-d_m) v_m = 0. \end{aligned}$$

But we know that $v_1, \dots, v_m, u_1, \dots, u_{n-m}$ are independent vectors. Therefore $c_1 = 0, \dots, c_{n-m} = 0, d_1 = 0, \dots, d_m = 0$. Consequently, $u = 0$. Therefore $U \cap null(T) = \{0\}$."

Now we want to show that $range(T) = \{T(u) | u \in U\}$. Since $U \subset V$, we have

$$\{T(u) | u \in U\} \subset \{T(v) | v \in V\} = range(T). \tag{9}$$

Conversely, let us choose $T(v) \in range(T)$. We want to show that $T(v) \in \{T(u) | u \in U\}$. Since $\{v_1, \dots, v_m, u_1, \dots, u_{n-m}\}$ is a basis of V, we can write

$$v = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_{n-m} u_{n-m}.$$

Therefore

$$\begin{aligned} T(v) &= a_1 T(v_1) + \cdots a_m T(v_m) + T(b_1 u_1 + \cdots + b_{n-m} u_{n-m}) \\ &= T(b_1 u_1 + \cdots + b_{n-m} u_{n-m}) \quad (\text{since } v_i \in null(T), \ \forall i = 1, \dots, m). \end{aligned}$$

But $b_1 u_1 + \cdots + b_{n-m} u_{n-m} \in U$, therefore $T(v) = T(b_1 u_1 + \cdots + b_{n-m} u_{n-m}) \in \{T(u) | u \in U\}$. Consequently,

$$range(T) \subset \{T(u) | u \in U\}. \tag{10}$$

Combining (9) and (10) we have the result

$$range(T) = \{T(u) | u \in U\}.$$

Remark: (1) Geometric idea: nullspace is annihilated by the linear transformation T. So the main contribution for range(T) is given by $V - null(T)$. To realize this fact, look at the construction of U.
(2) First part of the proof (which is included in "...") uses the similar technique of problem 5.6. But this time the technique is used in a reverse order. Both rely on the same geometric idea.

Problem 6.7: First of all, notice that $S : U \to V$, therefore $null(S)$ is a subspace of U. On the other hand $T \circ S : U \to W$, therefore $null(T \circ S)$ is a also subspace of U. But which one is bigger? Let $u \in null(S)$, then $T \circ S(u) = T(S(u)) = T(0_V) = 0_W$ (we are using the notation 0_X to indicate that it is the zero element of the vector space X). Which implies that $u \in null(T \circ S)$, therefore $null(S) \subset null(T \circ S)$.

Now let us say $dim(null(S)) = k$ and $\{s_1, \ldots, s_k\}$ be a basis of $null(S)$. We can extend it to a basis of $null(T \circ S)$, say $\{s_1, \ldots, s_k, u_1, \ldots, u_l\}$. Since each of $s_1, \ldots, s_k, u_1, \ldots, u_l$ is annihilated by $T \circ S$, either they are annihilated by S or their image (via S) is annihilated by T. We know that the vectors $s_1, \ldots, s_k$ are annihilated by S (because they belong to $null(S)$). Therefore images of $u_1, \ldots, u_l$ via S i.e., $S(u_1), \ldots, S(u_l)$ must be annihilated by T (because $T \circ S(u_i) = 0$). In other words

$$S(u_1), \ldots, S(u_l) \text{ belong to the } null(T). \tag{11}$$

But $S(u_1), \ldots, S(u_l)$ are l independent vectors. Because if

$$\begin{aligned} & c1S(u_1) + \cdots + c_l S(u_l) = 0 \\ i.e., \quad & S(c_1 u_1 + \cdots + c_l u_l) = 0. \end{aligned} \tag{12}$$

Then $c_1 u_1 + \cdots + c_l u_l \in null(S)$. But the $null(S)$ is spanned by another disjoint independent set of vectors, namely $s_1, \ldots, s_k$. Therefore $c_1 u_1 + \cdots + c_l u_l = 0$ (use the "$\ldots$" technique from problem 6.5 or the technique of problem 5.6). But $u_1, \ldots, u_l$ is a part of a basis (they are part of the basis of $null(T \circ S)$), therefore $u_1, \ldots, u_l$ are independent. Hence $c_1 = 0, \ldots, c_l = 0$. Then from (12) we can say that $S(u_1), \ldots, S(u_l)$ are independent.

Now we can revise the statement (11) and say that $null(T)$ contains l many independent vectors. Therefore $dim(null(T)) \geq l$. Recall that $\{s_1, \ldots, s_k, u_1, \ldots, u_l\}$ is a basis of $null(T \circ S)$. Therefore $dim(null(T \circ S)) = k + l$. Gluing all the statements together we have

$$dim(null(T \circ S)) = k + l = dim(null(S)) + l \leq dim(null(S)) + dim(null(T)).$$

<u>Alternative Proof (using the dimension formula)</u>

First of all, notice that $S : U \to V$, therefore $null(S)$ is a subspace of U. On the other hand $T \circ S : U \to W$, therefore $null(T \circ S)$ is a also subspace of U. But which one is bigger? Let $u \in null(S)$, then $T \circ S(u) = T(S(u)) = T(0_V) = 0_W$ (we are using the notation 0_X to indicate that it is the zero element of the vector space X). Which implies that $u \in null(T \circ S)$, therefore $null(S) \subset null(T \circ S)$.

Now let us say $dim(null(S)) = k$ and $\{s_1, \ldots, s_k\}$ be a basis of $null(S)$. We can extend it to a basis of $null(T \circ S)$, say $\{s_1, \ldots, s_k, u_1, \ldots, u_l\}$. Since each of $s_1, \ldots, s_k, u_1, \ldots, u_l$ is annihilated by $T \circ S$, either they are annihilated by S or their image (via S) is annihilated by T. We know that the vectors $s_1, \ldots, s_k$ are annihilated by S (because they belong to $null(S)$). Therefore images of $u_1, \ldots, u_l$ via S i.e., $S(u_1), \ldots, S(u_l)$ must be annihilated by T (because $T \circ S(u_i) = 0$). In other words

$$S(s_i) = 0_V \ \forall i = 1, \ldots, k \text{ and } S(u_1), \ldots, S(u_l) \text{ belong to the } null(T). \tag{13}$$

We know that $S : U \to V$, and $null(T \circ S) \subset U$. Let us restrict S to $null(T \circ S)$[1], then from (13) we can say that $range(S|_{null(T\circ S)}) \subset null(T)$. Therefore

$$dim\left(range(S|_{null(T\circ S)})\right) \leq dim(null(T)).$$

Now applying the dimension formula on $S|_{null(T\circ S)}$, and the above result we obtain

$$\begin{aligned} dim(null(T \circ S)) &= dim\left(null(S|_{null(T\circ S)})\right) + dim\left(range(S|_{null(T\circ S)})\right) \\ &= dim(null(S)) + dim\left(range(S|_{null(T\circ S)})\right) \\ &\leq dim(null(S)) + dim(null(T)). \end{aligned}$$

[1]Let $f : X \to Y$ be a function and $Z \subset X$. Restriction of f on Z is denoted by $f|_Z$. The domain of restricted $f|_Z$ is only Z (whereas originally f had a bigger domain, namely X).