MAT 67 **Homework 1**
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Problem 1.1 The given system of equations

$$ax_1 + bx_2 = 0$$
$$cx_1 + dx_2 = 0.$$

Multiplying the first equation by c and second equation by a we have

$$acx_1 + bcx_2 = 0$$
$$acx_1 + adx_2 = 0.$$

Now subtracting the first equation from the second equation we have $(ad - bc)x_2 = 0$. Now if $ad - bc \neq 0$ then we must have $x_2 = 0$. Substituting $x_2 = 0$ in the given equations we have

$$ax_1 = 0$$
$$cx_1 = 0.$$

Multiplying the first equation by d, second by b and subtracting the second equation from the first one we have $(ad - bc)x_1 = 0$. But $ad - bc \neq 0$, therefore $x_1 = 0$. Consequently, $x_1 = 0, x_2 = 0$ is the only solution of the given system of equations.

Remark: Converse of the above statement is also true.

Conversely, suppose $x_1 = 0 = x_2$ is the only solution. We want to show that $ad - bc \neq 0$. Suppose if possible $ad - bc = 0$, then we have two cases.
Case 1: $ad - bc = 0$ and all of $a, b, c, d = 0$. Then it is easy to see that $x_1 = 1 = x_2$ is also a solution of the system. Which contradicts the fact that '$x_1 = 0 = x_2$ is the only solution'.
Case 2: $ad - bc = 0$ and at least one of $a, b, c, d \neq 0$. Without loss of generality we can assume that $a \neq 0$. Then we can verify that $x_1 = -b, x_2 = a$ is also a solution. Which again contradicts the fact that '$x_1 = 0 = x_2$ is the only solution'.
Therefore we can conclude that if $x_1 = 0 = x_2$ is the only solution then we must have $ad - bc = 0$.

Problem 2.1 Let us write $z = x + iy, w = u + iv$, where $x, y, u, v \in \mathbb{R}$.

(a) Then $az = ax + iay$. Since $a \in \mathbb{R}$, clearly we have $Re(az) = ax = aRe(z)$ and $Im(az) = ay = aIm(z)$.

(b) $z+w = x+iy+u+iv = (x+u)+i(y+v)$. Therefore $Re(z+w) = x+u = Re(z)+Re(w)$ and $Im(z + w) = y + v = Im(z) + Im(w)$.

Remark: If $a \notin \mathbb{R}$ then *Problem* 2.1(a) may not be true. For example take $z = 2+3i$ and $a = 1+2i$. Then $az = -4+7i$. Therefore $Re(az) = -4$, whereas $aRe(z) = (1+2i)2 = 2+4i$. Clearly $Re(az) \neq aRe(z)$.

Problem 2.3 Let us write $z = x+iy, w = u+iv$, where $x, y, u, v \in \mathbb{R}$. Then $z + w = (x+u) + i(y+v)$ and $z - w = (x-u) + i(y-v)$. Now we notice that

$$\begin{aligned} |z|^2 &= x^2 + y^2 \\ |w|^2 &= u^2 + v^2 \\ |z-w|^2 &= (x-u)^2 + (y-v)^2 \\ |z+w|^2 &= (x+u)^2 + (y+v)^2. \end{aligned}$$

Therefore

$$\begin{aligned} |z-w|^2 + |z+w|^2 &= (x-u)^2 + (y-v)^2 + (x+u)^2 + (y+v)^2 \\ &= [(x-u)^2 + (x+u)^2] + [(y-u)^2 + (y+v)^2] \\ &= 2(x^2+u^2) + 2(y^2+v^2) \\ &= 2(x^2+y^2+u^2+v^2) \\ &= 2(|z|^2 + |w|^2). \end{aligned}$$

Alternative Solution

We know that if u is a complex number, then $|u|^2 = u\bar{u}$. Using that property we have

$$\begin{aligned} |z-w|^2 + |z+w|^2 &= (z-w)(\overline{z-w}) + (z+w)(\overline{z+w}) \\ &= (z-w)(\bar{z}-\bar{w}) + (z+w)(\bar{z}+\bar{w}) \\ &= (z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w}) + (z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}) \\ &= 2(z\bar{z} + w\bar{w}) \\ &= 2(|z|^2 + |w|^2). \end{aligned}$$

Problem 2.4 We know that either $|z| = 1$ or $|w| = 1$. Let us assume $|z| = 1$. Then we can write $z = e^{i\theta}$ and $w = re^{i\phi}$, where $r, \theta, \phi \in \mathbb{R}$ (and r is not necessarily 1). Now we can compute

$$\begin{aligned} \left|\frac{z-w}{1-\bar{z}w}\right| &= \left|\frac{e^{i\theta} - re^{i\phi}}{1 - e^{-i\theta}re^{i\phi}}\right| \\ &= \left|\frac{e^{i\theta}\left(1 - re^{i(\phi-\theta)}\right)}{1 - re^{i(\phi-\theta)}}\right| \\ &= |e^{i\theta}| \quad (\text{since } \bar{z}w = re^{i(\phi-\theta)} \neq 1) \\ &= 1. \end{aligned}$$

case 2: Assume $|w| = 1$. Computation is similar as above.

Alternative Solution

Let us write

$$a = \frac{z - w}{1 - \bar{z}w}.$$

Then

$$\bar{a} = \frac{\overline{z - w}}{\overline{1 - \bar{z}w}} = \frac{\bar{z} - \bar{w}}{1 - z\bar{w}}.$$

So we can compute

$$\begin{aligned} a\bar{a} &= \frac{z - w}{1 - \bar{z}w}\frac{\bar{z} - \bar{w}}{1 - z\bar{w}} \\ &= \frac{(z - w)(\bar{z} - \bar{w})}{(1 - \bar{z}w)(1 - z\bar{w})} \\ &= \frac{z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w}}{1 - z\bar{w} - \bar{z}w + \bar{z}wz\bar{w}} \\ &= \frac{|z|^2 + |w|^2 - (z\bar{w} + w\bar{z})}{1 + |z|^2|w|^2 - (z\bar{w} + w\bar{z})}. \end{aligned}$$

Now if $|z| = 1$, then from the last equation we have

$$a\bar{a} = \frac{1 + |w|^2 - (z\bar{w} + w\bar{z})}{1 + |w|^2 - (z\bar{w} + w\bar{z})} = 1.$$

Similarly if $|w| = 1$,

$$a\bar{a} = \frac{|z|^2 + 1 - (z\bar{w} + w\bar{z})}{1 + |z|^2 - (z\bar{w} + w\bar{z})} = 1.$$

So in any case $a\bar{a} = 1$ i.e., $|a|^2 = 1$. Therefore $|a| = 1$ (since $|a|$ can not be -1). Which implies that

$$\left|\frac{z - w}{1 - \bar{z}w}\right| = 1.$$

Problem 2.5 Let $z = x + iy = (x, y)$ be a complex number. If we rotate z counter-clockwise by an angle θ, then we will get a new complex number given by $ze^{i\theta}$. Therefore basically the map f_θ is nothing but $f_\theta(z) = ze^{i\theta}$. But this f_θ is $f_\theta : \mathbb{C} \to \mathbb{C}$, whereas we want to write f_θ as $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$.

We notice that vector representation of $z = (x, y)$. On the other hand $ze^{i\theta} = (x + iy)(\cos\theta + i\sin\theta) = (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$. Therefore vector representation of $ze^{i\theta} = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. So finally the $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Problem 3.2

(a) Following the definition of $\bar{p}(z)$ we have

$$\bar{p}(z) = \bar{a}_n z^n + \cdots + \bar{a}_1 z + \bar{a}_0.$$

Therefore

$$\begin{aligned} \bar{p}(\bar{z}) &= \bar{a}_n \bar{z}^n + \cdots + \bar{a}_1 \bar{z} + \bar{a}_0 \\ &= \overline{a_n z^n + \cdots + a_1 z + a_0} \\ &= \overline{p(z)}. \end{aligned}$$

(b) Suppose $p(z)$ has real coefficients i.e., a_i s are real for all $i = 0, \ldots, n$. Then $\bar{a}_i = a_i$ for all $i = 0, \ldots, n$. Which implies that

$$\begin{aligned} \bar{p}(z) &= \bar{a}_n z^n + \cdots + \bar{a}_1 z + \bar{a}_0 \\ &= a_n z^n + \cdots + a_1 z + a_0 \quad (\text{since } \bar{a}_i = a_i, \ \forall\, i = 0, \ldots, n) \\ &= p(z). \end{aligned}$$

Conversely, suppose $\bar{p}(z) = p(z)$. Then we have

$$\bar{a}_n z^n + \cdots + \bar{a}_1 z + \bar{a}_0 = a_n z^n + \cdots + a_1 z + a_0.$$

Comparing the coefficients of z^i from both sides of the above equation we have $\bar{a}_n = a_n, \bar{a}_{n-1} = a_{n-1}, \ldots, \bar{a}_1 = a_1, \bar{a}_0 = a_0$. Therefore all a_i s (for $i = 0, \ldots, n$) i.e., the coefficients of $p(z)$ are real numbers.

(c) *[Basically this problem is using the property that $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ for any two complex numbers z_1 and z_2]* Let us take

$$\begin{aligned} q(z) &= a_m z^m + \cdots + a_1 z + a_0 \\ r(z) &= b_n z^n + \cdots + b_1 z + b_0. \end{aligned}$$

Now $p(z) = q(z)r(z)$ is a $m + n$ degree polynomial. We can write

$$p(z) = a_m b_n z^{m+n} + \cdots + (a_1 b_0 + b_1 a_0) z + a_0 b_0.$$

Therefore

$$\begin{aligned} \bar{p}(z) &= \overline{a_m b_n} z^{m+n} + \cdots + (\overline{a_1 b_0 + b_1 a_0}) z + \overline{a_0 b_0} \\ &= \bar{a}_m \bar{b}_n z^{m+n} + \cdots + (\bar{a}_1 \bar{b}_0 + \bar{b}_1 \bar{a}_0) z + \bar{a}_0 \bar{b}_0. \end{aligned}$$

Now

$$\bar{q}(z) = \bar{a}_m z^m + \cdots + \bar{a}_1 z + \bar{a}_0$$
$$\bar{r}(z) = \bar{b}_n z^n + \cdots + \bar{b}_1 z + \bar{b}_0.$$

Therefore

$$\begin{aligned}\bar{q}(z)\bar{r}(z) &= \bar{a}_m \bar{b}_n z^{m+n} + \cdots + (\bar{a}_1 \bar{b}_0 + \bar{b}_1 \bar{a}_0) z + \bar{a}_0 \bar{b}_0 \\ &= \bar{p}(z).\end{aligned}$$

Alternative Solution

We know that $p(z) = q(z)r(z)$. Multiplying both sides by $\bar{q}(z)\bar{r}(z)$ we have

$$\begin{aligned}p(z)\bar{q}(z)\bar{r}(z) &= q(z)r(z)\bar{q}(z)\bar{r}(z) \\ &= |q(z)|^2 |r(z)|^2.\end{aligned}$$

Consequently,

$$[p(z)]^{-1} = \frac{\bar{q}(z)\bar{r}(z)}{|q(z)|^2 |r(z)|^2}.$$

But we know that if u is a complex number then $u^{-1} = \frac{\bar{u}}{|u|^2}$. In our case, considering $p(z)$ as u, we have $\bar{u} = \bar{q}(z)\bar{r}(z)$ i.e., $\overline{p(z)} = \bar{q}(z)\bar{r}(z)$.

Problem 3.3: Let

$$p(z) = a_n z^n + \cdots + a_1 z + a_0 \tag{1}$$

be a polynomial with real coefficients i.e., a_i s are real numbers for all $i = 0, \ldots, n$. Since $p(\alpha) = 0$, we have

$$a_n \alpha^n + \cdots + a_1 \alpha + a_0 = 0.$$

Now taking conjugate on both sides we have

$$\begin{aligned} &\overline{a_n \alpha^n + \cdots + a_1 \alpha + a_0} = 0 \\ i.e., \quad &\bar{a}_n \bar{\alpha}^n + \cdots + \bar{a}_1 \bar{\alpha} + \bar{a}_0 = 0 \\ i.e., \quad &a_n \bar{\alpha}^n + \cdots + a_1 \bar{\alpha} + a_0 = 0 \quad \text{(since } a_i\text{s are real).} \end{aligned} \tag{2}$$

Now plugging in $z = \bar{\alpha}$ in (1) we have

$$p(\bar{\alpha}) = a_n \bar{\alpha}^n + \cdots + a_1 \bar{\alpha} + a_0.$$

Using (2) we conclude that $p(\bar{\alpha}) = 0$.

Remark: The above result says that if $p(z)$ is a polynomial with real coefficients then complex roots occur in conjugate pairs. Which also implies that there are even number of complex roots.