MAT 67 Max time: 1 hr 30 mins **Final Exam**

Name:____ September 11, 2014

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- Check that you have all seven pages of the exam. (The number of pages includes this cover page.)
- There is an *extra credit problem* in the last page.

During the exam:

- Keep your eyes on your own exam!
- No notes/books or electronics AT ALL!

Note that the exam length is exactly 1 hr 30 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	**Total**
Score					

Problem 1:*(20 points)* TRUE or FALSE? If the statement is true then give a proof. If the statement is false then give a counterexample.

(a)(5 points) Let $v_1, v_2, v_3 \in \mathbb{R}^3$. Then $span\{v_1, v_2, v_3\} = \mathbb{R}^3$.

Solution: FALSE. For example take $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (1, 1, 0)$. These are dependent vectors and $span\{v_1, v_2, v_3\} \neq \mathbb{R}^3$.

(b)(5 points) The vector $v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$.

Solution: TRUE. We can verify that

$$Av = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 5v.$$

Therefore v is an eigenvector of A, and the corresponding eigenvalue is 5.

(c)(5 points) The matrix $M = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is <u>not</u> invertible.

Solution: FALSE. Notice that $\det(A) = -3 \neq 0$. Therefore A is invertible.

(d)(5 points) A linear map $T : \mathbb{R}^2 \to \mathbb{R}^4$ can <u>not</u> be surjective (i.e., onto).

Solution: TRUE. According to the dimension formula

$$\begin{aligned} & \dim(\mathbb{R}^2) = \dim(null(T)) + \dim(Range(T)) \\ \Rightarrow \quad & 2 = \dim(null(T)) + \dim(Range(T)). \end{aligned}$$

If T is surjective, then $Range(T) = \mathbb{R}^4$ i.e., $\dim(Range(T)) = 4$. Then from the above formula we have $\dim(null(T)) = -2$. But dimension can not be negative - a contradiction.

Problem 2: *(20 points)* Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

(a) *(2 points)* The matrix A can be considered as a linear map $A : \mathbb{R}^p \to \mathbb{R}^q$. What are the values of p and q?

(b) *(3 points)* Define the null space of the linear transformation A.

(c) *(8 points)* Solve the system of equations

$$\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(d) *(3 points)* What is a basis of $null(A)$?

(e) *(2 points)* Then what is the dimension of the null space of A?

(f) *(2 points)* Therefore the dimension of $Range(A)$ is ______.

Solution:

(a) A can be considered as a linear map $A : \mathbb{R}^4 \to \mathbb{R}^2$ i.e., $p = 4, q = 2$.

(b) The null space of A is

$$null(A) := \{x \in \mathbb{R}^4 : Ax = 0\}.$$

(c) In the given system of equations, we notice that x_1, x_3 are pivot variables and x_2, x_4 are free variables. Since there are two free variables, there will be two independent solutions.

<u>*Sol 1:*</u> Assign $x_2 = 1, x_4 = 0$. Then given system of equations become

$$\begin{aligned} x_1 + 2 - x_3 &= 0 \\ 2x_3 &= 0. \end{aligned}$$

Solving the above system we have $x_3 = 0, x_1 = -2$. Therefore $(-2, 1, 0, 0)$ is a solution of the given system of equations.

<u>*Sol 2:*</u> Assign $x_2 = 0, x_4 = 1$. In this case the given system becomes as

$$\begin{aligned} x_1 - x_3 - 2 &= 0 \\ 2x_3 + 4 &= 0. \end{aligned}$$

Solving the above system we obtain $x_3 = -2, x_1 = 0$. Therefore $(0, 0, -2, 1)$ is also a solution of the system.

(d) From the *part (c)* we can say that a basis of $null(A)$ is $\{(-2, 1, 0, 0), (0, 0, -2, 1)\}$.

(e) Since there are two vectors in the basis, we have $\dim(null(A)) = 2$.

(f) Using the dimension formula, we have $\dim(Range(A)) = \dim(\mathbb{R}^4) - \dim(null(A)) = 2$.

Problem 3: *(20 points)[Projection Matrix]* Consider the vectors $(1,0,1),(1,0,-1) \in \mathbb{R}^3$.

(a) *(1 points)* Construct a 3×2 matrix M such that the columns of M are the above two vectors.

(b) *(1 points)* Write down that the transpose of the matrix M. Check that M^T is a 2×3 matrix.

(c) *(2 points)* Compute the product M^TM. Check that it is a 2×2 matrix.

($\star$) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an invertible matrix then A^{-1} is given by the formula $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(d) *(3 points)* Using the formula ($\star$) or otherwise, compute the inverse of M^TM.

(e) *(3 points)* Using your results from *part (d)* and *part (b)*, calculate $(M^TM)^{-1}M^T$. Check that it is a 2×3 matrix.

(f) *(3 points)* Using your results from *part (e)* and *part (a)*, calculate $M(M^TM)^{-1}M^T$. Check that it is a 3×3 matrix.

(g) *(3 points)* Verify that the original vectors $(1,0,1),(1,0,-1)$ are the eigenvectors of the final matrix $M(M^TM)^{-1}M^T$. What are the corresponding eigenvalues?

(h) *(2 points)* Compute the determinant of $M(M^TM)^{-1}M^T$. Then what is the other eigenvalue?

(i) *(2 points)* If $P = M(M^TM)^{-1}M^T$ for some matrix M, then prove that $P^2 = P$ *(such a matrix P is called a projection matrix).*

Solution:

(a) The matrix M is

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

(b) The transpose of M is given by

$$M^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

(c) The product

$$M^TM = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(d) Using the formula ($\star$) we have found that

$$(M^TM)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(e)

$$(M^TM)^{-1}M^T = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

(f)

$$M(M^TM)^{-1}M^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(g) We can vrify that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Therefore the vectors $(1, 0, 1)$ and $(1, 0, -1)$ are the eigenvectors of $M(M^TM)^{-1}M^T$, and the corresponding eigenvalues are $\lambda_1 = 1, \lambda_2 = 1$.

(h) It is easy to see that $\det[M(M^TM)^{-1}M^T] = 0$. Therefore $\lambda_3 = 0$ is the other eigenvalue.

(i) If $P = M(M^TM)^{-1}M^T$, then we have

$$\begin{aligned} P^2 &= M(M^TM)^{-1}M^TM(M^TM)^{-1}M^T \\ &= M(M^TM)^{-1}(M^TM)(M^TM)^{-1}M^T \\ &= M(M^TM)^{-1}M^T \\ &= P. \end{aligned}$$

Problem 4: *(20 points)* Consider the vector space $\mathbb{R}_2[x] = \{polynomials\ with\ real\ coefficients\ of\ degree\ at\ most\ two\}$. We define the inner product between two polynomials $p(x)$ and $q(x)$ by

$$\langle p, q \rangle := \int_0^1 p(x)q(x)\, dx. \tag{1}$$

Notice that the set $\{1, x, x^2\}$ forms a basis of $\mathbb{R}_2[x]$ *(you do not need to prove this statement).*

(a) *(6 points)* Using the definition (1), compute the inner products between the basis vectors i.e., compute $\langle 1, x\rangle, \langle 1, x^2\rangle, \langle x, x^2\rangle$.

(b) *(2 points)* Are these basis vectors orthogonal to each other?

(c) *(12 points)* Use the Gram-Schmidt orthogonalization method to find an orthogonal basis of $\mathbb{R}_2[x]$ *(you do not need to normalize the final vectors).*

Solution:

(a) Using the given definition of the inner product we have

$$\begin{aligned}
\langle 1, x\rangle &= \int_0^1 x\, dx = \frac{1}{2} \\
\langle 1, x^2\rangle &= \int_0^1 x^2\, dx = \frac{1}{3} \\
\langle x, x^2\rangle &= \int_0^1 x \cdot x^2\, dx = \int_0^1 x^3\, dx = \frac{1}{4}.
\end{aligned}$$

(b) Since the inner products between the basis vectors are not zero, they are not orthogonal to each other.

(c) Let us call $f_1 = 1, f_2 = x, f_3 = x^2$. Using the Gram-Schmidt orthogonalization

$$\begin{aligned}
\tilde{f}_1 &= 1 \\
\tilde{f}_2 &= f_2 - \frac{\langle f_2, \tilde{f}_1\rangle}{\langle \tilde{f}_1, \tilde{f}_1\rangle}\tilde{f}_1 \\
&= x - \frac{\int_0^1 x\, dx}{\int_0^1 1\, dx}1 \\
&= x - \frac{1/2}{1}1 \\
&= x - \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_3 &= f_3 - \frac{\langle f_3, \tilde{f}_2\rangle}{\langle \tilde{f}_2, \tilde{f}_2\rangle}\tilde{f}_2 - \frac{\langle f_3, \tilde{f}_1\rangle}{\langle \tilde{f}_1, \tilde{f}_1\rangle}\tilde{f}_1 \\
&= x^2 - \frac{\int_0^1 x^2(x - 1/2)\, dx}{\int_0^1 (x - 1/2)^2\, dx}\left(x - \frac{1}{2}\right) - \frac{\int_0^1 x^2\, dx}{\int_0^1 1\, dx}1 \\
&= x^2 - \frac{1/4 - 1/6}{1/12}\left(x - \frac{1}{2}\right) - \frac{1/3}{1}1 \\
&= x^2 - \left(x - \frac{1}{2}\right) - \frac{1}{3} \\
&= x^2 - x + \frac{1}{6}.
\end{aligned}$$

. Therefore an orthogonal basis of $\mathbb{R}_2[x]$ is $\{1, (x - 1/2), x^2 - x + 1/6\}$.

Extra Credit: *(2 points)* Write/draw something mathematical/non-mathematical .