MAT 21D **Midterm Review**

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0. General Tips

- It is recommended to change the limits of integration while doing a substitution.
- First write the main formula (eg. *centroid, moment of inertia, mass, work flow, etc.*) then put the limits of the integration.
- Drawing a picture is recommended. Pictures of the curve/area/solid will help you to choose the right coordinate system and to find the limits of the integration.
- Observe the symmetry of both the *solid* and the *density* before attempting to find the centroid. Symmetry eliminates unnecessary calculations and helps to choose the right coordinate system.
- Always write the '$\rightarrow$' sign on top of a vector quantity.

1. Chapter 15

Double Integration

- Draw the region of integration. This will help you to find the limits of integration accurately.
- If you have $\int_a^b \int_{x=g_1(y)}^{g_2(y)} f(x,y)\ dxdy$, figure out the limits of x first (by holding y at a generic position), then figure out the limits of y. For $dydx$, it is other way around.
- If the integration seems impossible to compute, try reversing the order of integration or try converting it to the polar coordinate system.

Triple Integration:

- If you have $\int_a^b \int_{y=g_1(x)}^{g_2(x)} \int_{z=h_1(x,y)}^{h_2(x,y)} f(x,y,z)\ dzdydx$, figure out the limits of z first (by holding x, y at a generic position). Then take the shadow of the solid onto the xy-plane and figure out the limits of y and then limits of x. If the integration is in a different order eg. $dxdydz$, then do x first, take the shadow onto yz-plane and figure out the limits of y and z.
- If the solid is cylindrically symmetric i.e., the base of the solid is a part of circle or an ellipse, often it is convenient to use the cylindrical coordinate system. For example, consider the solid (D) bounded by the cylinder $y^2 + z^2 = 9$, the yz-plane, and the paraboloid $x = y^2 + z^2$. This solid is based on the yz-plane and sticks out of yz-plane towards the positive x axis. It is better to use $dx\ rdrd\theta$ order of integration i.e., $\int\int\int_D f(x,y,z)\ dV$ will become

$$\int_0^{2\pi} \int_0^3 \int_0^{r^2} f(x, r\cos\theta, r\sin\theta)\ dx\ rdrd\theta. \tag{1}$$

- *Relations between the rectangular, cylindrical, and the spherical coordinate system:* If (x, y, z) is the coordinate of a point P in the rectangular coordinate system, (r, θ, z) is the coordinate of the same point in the cylindrical system, and (ρ, ϕ, θ) in the spherical system then

$$\begin{aligned} x &= r\cos\theta & x &= \rho\sin\phi\cos\theta \\ y &= r\sin\theta & y &= \rho\sin\phi\sin\theta \\ z &= z & z &= \rho\cos\phi. \end{aligned}$$

  The cylindrical coordinates are not always represented by (r, θ, z). Sometimes, depending on the orientation of the solid, (x, r, θ) or (r, y, θ) are convenient. For example, (x, r, θ) was used in (1). Similar argument is true for spherical coordinate system too.

- If the solid is a part of a sphere or an ellipsoid, then it is better to use the spherical coordinate system. For example, consider the solid (D) trapped inside the sphere $x^2 + y^2 + z^2 = 7$ and outside of the cone $z = \pm\sqrt{x^2 + y^2}$. Then $\int\int\int_D f(x, y, z)\, dV$ becomes

$$\int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^{\sqrt{7}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho d\phi d\theta.$$

Mass, Centroid, Moment of Inertia: Let $\delta(x, y, z)$ be the density of a solid D.

- Mass= $M = \int\int\int_D \delta(x, y, z)\, dV$, where $dV = dzdydx = dz\, rdrd\theta = \rho^2\sin\phi\, d\rho d\phi d\theta$ depending on the chosen coordinate system.

- First moments:

$$M_{yz} = \int\int\int_D x\delta(x, y, z)\, dV \quad M_{xz} = \int\int\int_D y\delta(x, y, z)\, dV \quad M_{xy} = \int\int\int_D z\delta(x, y, z)\, dV.$$

- Centroid: $(\bar{x}, \bar{y}, \bar{z}) = (M_{yz}/M, M_{xz}/M, M_{xy}/M)$.

- Moment of Inertia: Let L be the axis of rotation, and $d(x, y, z)$ be the distance of a generic point (x, y, z) from L. Then the moment of inertia of the solid D with respect to the line L is

$$I_L = \int\int\int_D d(x, y, z)^2\delta(x, y, z)\, dV. \tag{2}$$

  For example, if L is the $x-$axis, then (2) becomes

$$I_x = \int\int\int_D (y^2 + z^2)\delta(x, y, z)\, dV.$$

  If L is the line parallel to z-axis and passing through the point $(1, 2, -1)$ then (2) becomes

$$I_L = \int\int\int_D \left[(x-1)^2 + (y-2)^2\right]\delta(x, y, z)\, dV.$$

2. Chapter 13

Here is the main framework

- Curve: $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$.

- Velocity: $\vec{v}(t) = \frac{d\vec{r}}{dt} = x'(t)\,\hat{i} + y'(t)\,\hat{j} + z'(t)\,\hat{k}$.

- Speed: $|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.
- Unit tangent vector: $\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$.
- Arc length parameter: $s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$, where t_0 is the initial time and t is the final time.
- $\frac{ds}{dt} = |\vec{v}(t)|$.
- Arc length: $\int_a^b |\vec{v}(t)|\, dt$, where $t = a$ is the initial time and $t = b$ is the final time.
- Curvature: $\kappa = \frac{|d\vec{T}/dt|}{|\vec{v}(t)|}$.
- Unit normal vector: $\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$.
- Binormal vector: $\vec{B} = \vec{T} \times \vec{N}$.
- Acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$.
- Tangential and normal components of acceleration: $\vec{a} = a_T\vec{T} + a_N\vec{N}$, where the tangential component is $a_T = \frac{d|\vec{v}|}{dt}$ and the normal component is $a_N = \kappa|\vec{v}|^2 = \sqrt{|\vec{a}|^2 - a_T^2}$.

3. Chapter 16

There are mainly two types of line integrals, Line integral of a scalar field and line integral in a vector field.

(1) Line integral of a scalar function: This is used to compute the area bounded above a curve and below a surface (see figure 16.5). This can also be used to compute the mass, centroid, and moment of inertia of a wire. Let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k};\ \ a \le t \le b$ be the equation of a space curve, and $f(x, y, z)$ be a scalar function then the line integral of $f(x, y, z)$ over the curve is given by

$$\int_a^b f(x(t), y(t), z(t))|\vec{v}(t)|\, dt. \tag{3}$$

If $\vec{r}(t)$ is a planer curve (eg. $\vec{r}(t) = y(t)\hat{\jmath} + z(t)\hat{k}$ is a planer curve on the yz-plane), and f is a function defined on the same plane (eg. $f(y, z)$ is a function defined on the yz-plane), then (3) gives the area of the wall bounded in between the curve and the function f (similar to figure 16.5).

If $f(x, y, z) = \delta(x, y, z)$ is the density of the wire, the (3) gives the mass of the wire.

If $f(x, y, z) = (x^2 + y^2)\delta(x, y, z)$, where $\delta(x, y, z)$ is the density of the wire, then (3) gives the moment of inertia of the wire with respect to the z-axis etc. See the Table 16.1 for a complete set of formula.

(2) Line integral in a vector field: This is used to compute the work, flow, circulation, flux. Let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k};\ \ a \le t \le b$ be a space curve, and $\vec{F}(x, y, z) = M(x, y, z)\hat{\imath} + N(x, y, z)\hat{\jmath} + P(x, y, z)\hat{k}$ be a vector field.

(a) *Work:* The work done while moving a particle from the point $\vec{r}(a)$ to the point $\vec{r}(b)$ is given by

$$\int_a^b \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt}\, dt = \int_a^b (M\ dx + N\ dy + P\ dz). \tag{4}$$

There is a list of equivalent formula given in the Table 16.2.

(b) *Flow:* Flow of the vector field $\vec{F}$ along the curve $\vec{r}(t)$ from the point $\vec{r}(a)$ to the point $\vec{r}(b)$ is given by

$$\int_a^b \vec{F} \cdot d\vec{r}. \tag{5}$$

Note that mathematically (4) and (5) are the same.

If the curve starts and ends at the same point i.e., $\vec{r}(a) = \vec{r}(b)$, then the flow is called the *circulation* of the vector field along the curve.

(c) *Flux:* Flux of a vector field $\vec{F}(x,y) = M(x,y)\ \hat{\imath} + N(x,y)\ \hat{\jmath}$ across a simple closed curve $C \equiv \vec{r}(t) = x(t)\ \hat{\imath} + y(t)\ \hat{\jmath}$ is given by

$$\oint_C (M\ dy - N\ dx).$$

Conservative Vector Field:

- A vector field $\vec{F}(x,y,z) = M(x,y,z)\hat{\imath} + N(x,y,z)\hat{\jmath} + P(x,y,z)\hat{k}$ is conservative

$$\begin{aligned}
&\equiv \quad \text{There exists a potential function } f \text{ such that } \vec{F} = \vec{\nabla} f = \frac{\partial f}{\partial x}\ \hat{\imath} + \frac{\partial f}{\partial y}\ \hat{\jmath} + \frac{\partial f}{\partial z}\ \hat{k}.\\
&\equiv \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x,y,z) & N(x,y,z) & P(x,y,z) \end{vmatrix} = 0.\\
&\equiv \quad \oint_C \vec{F} \cdot d\vec{r} = 0, \text{ for any simple closed curve } C.\\
&\equiv \quad \int_A^B \vec{F} \cdot d\vec{r} \text{ is independent of the path which connects the two points } A \text{ and } B
\end{aligned} \tag{6}$$

['$\equiv$' indicates that the statements are equivalent].

- <u>*Finding the Potential Function*</u> Let $f(z,y,z)$ be the potential function of the vector field $\vec{F}(x,y,z) = M(x,y,z)\hat{\imath} + N(x,y,z)\hat{\jmath} + P(x,y,z)\hat{k}$. Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= M(x,y,z) & &\Rightarrow f(x,y,z) = \int M(x,y,z)\ dx + C_1(y,z)\\
\frac{\partial f}{\partial y} &= N(x,y,z) & &\Rightarrow f(x,y,z) = \int N(x,y,z)\ dy + C_2(x,z)\\
\frac{\partial f}{\partial z} &= P(x,y,z) & &\Rightarrow f(x,y,z) = \int P(x,y,z)\ dz + C_3(x,y).
\end{aligned}$$

To find the $f(x,y,z)$, add the above three equations *but ignore the repeated expressions.* For example, if

$$\int M(x,y,z)\ dx = e^x \cos y + xyz \tag{7}$$

$$\int N(x,y,z)\ dy = xyz + e^x \cos y \tag{8}$$

$$\int P(x,y,z)\ dz = xyz + \frac{z^2}{2}, \tag{9}$$

then

$$f(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C. \tag{10}$$

Note that $e^x \cos y$ was present in the equations (7) and (8) but it was written only once in (10). Similarly, xyz was present in all of (7), (8), and (9), but it was written only once in (10).

Alternatively, you may use the method described in the book.

- If a particle is moving from the point A to the point B in a conservative force field, then the work done is independent of the path and by the fundamental theorem of line integral, (4) and (6) become

$$\text{Work} = \int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A),$$

where f is the potential function of $\vec{F}$.