MAT 21D **Review**

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0. General Tips

- It is recommended to change the limits of integration while doing a substitution.
- First write the main formula (eg. *centroid, moment of inertia, mass, work flow, etc.*) then put the limits of the integration.
- Drawing a picture is recommended. Pictures of the curve/area/solid will help you to choose the right coordinate system and to find the limits of the integration.
- Observe the symmetry of both the *solid* and the *density* before attempting to find the centroid. Symmetry eliminates unnecessary calculations and helps to choose the right coordinate system.
- Always write the '$\rightarrow$' sign on top of a vector quantity.
- Note that $\frac{\partial}{\partial x}$ represents the partial derivative with respect to x, and $\frac{d}{dx}$ represents the derivative with respect to x. These notations should be used properly.

1. Chapter 15

Double Integration

- Draw the region of integration. This will help you to find the limits of integration accurately.
- If you have $\int_a^b \int_{x=g_1(y)}^{g_2(y)} f(x,y)\ dxdy$, figure out the limits of x first (by holding y at a generic position), then figure out the limits of y. For $dydx$, it is other way around.
- If the integration seems impossible to compute, try reversing the order of integration or try converting it to the polar coordinate system.
- There is always an extra 'r' factor in the integration in polar coordinate system, namely $r\ drd\theta$.

Triple Integration:

- If you have $\int_a^b \int_{y=g_1(x)}^{g_2(x)} \int_{z=h_1(x,y)}^{h_2(x,y)} f(x,y,z)\ dzdydx$, figure out the limits of z first (by holding x, y at a generic position). Then take the shadow of the solid onto the xy-plane and figure out the limits of y and then limits of x. If the integration is in a different order eg. $dxdydz$, then do x first, take the shadow onto yz-plane and figure out the limits of y and z.
- If the solid is cylindrically symmetric i.e., the base of the solid is a part of circle or an ellipse, often it is convenient to use the cylindrical coordinate system. For example, consider the solid (D) bounded by the cylinder $y^2 + z^2 = 9$, the yz-plane, and the paraboloid $x = y^2 + z^2$. This solid is based on the yz-plane and sticks out of yz-plane towards the positive x axis. It is better to use $dx\ rdrd\theta$ order of integration i.e., $\iiint_D f(x,y,z)\ dV$ will become

$$\int_0^{2\pi} \int_0^3 \int_0^{r^2} f(x, r\cos\theta, r\sin\theta)\ dx\ rdrd\theta. \tag{1}$$

- *Relations between the rectangular, cylindrical, and the spherical coordinate system:* If (x, y, z) is the coordinate of a point P in the rectangular coordinate system, (r, θ, z) is the coordinate of the same point in the cylindrical system, and (ρ, ϕ, θ) in the spherical system then

$$\begin{aligned} x &= r\cos\theta & x &= \rho\sin\phi\cos\theta \\ y &= r\sin\theta & y &= \rho\sin\phi\sin\theta \\ z &= z & z &= \rho\cos\phi. \end{aligned}$$

  The cylindrical coordinates are not always represented by (r, θ, z). Sometimes, depending on the orientation of the solid, (x, r, θ) or (r, y, θ) are convenient. For example, (x, r, θ) was used in (1). Similar argument is true for spherical coordinate system too.

- There is always extra 'r' factor in the integration in cylindrical coordinate system (i.e., $r\ dzdrd\theta$), and extra '$\rho^2 \sin\phi$' in the spherical coordinate system (i.e., $\rho^2 \sin\phi\ d\rho d\phi d\theta$).

- If the solid is a part of a sphere or an ellipsoid, then it is better to use the spherical coordinate system. For example, consider the solid (D) trapped inside the sphere $x^2 + y^2 + z^2 = 7$ and outside of the cone $z = \pm\sqrt{x^2 + y^2}$. Then $\iiint_D f(x, y, z)\ dV$ becomes

$$\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^{\sqrt{7}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\ d\rho d\phi d\theta.$$

Mass, Centroid, Moment of Inertia: Let $\delta(x, y, z)$ be the density of a solid D.

- Mass= $M = \iiint_D \delta(x, y, z)\ dV$, where $dV = dzdydx = dz\ rdrd\theta = \rho^2 \sin\phi\ d\rho d\phi d\theta$ depending on the chosen coordinate system.

- First moments:

$$M_{yz} = \iiint_D x\delta(x, y, z)\ dV \quad M_{xz} = \iiint_D y\delta(x, y, z)\ dV \quad M_{xy} = \iiint_D z\delta(x, y, z)\ dV.$$

- Centroid: $(\bar{x}, \bar{y}, \bar{z}) = (M_{yz}/M, M_{xz}/M, M_{xy}/M)$.

- Moment of Inertia: Let L be the axis of rotation, and $d(x, y, z)$ be the distance of a generic point (x, y, z) from L. Then the moment of inertia of the solid D with respect to the line L is

$$I_L = \iiint_D d(x, y, z)^2 \delta(x, y, z)\ dV. \tag{2}$$

  For example, if L is the $x-$axis, then (2) becomes

$$I_x = \iiint_D (y^2 + z^2)\delta(x, y, z)\ dV.$$

  If L is the line parallel to z-axis and passing through the point $(1, 2, -1)$ then (2) becomes

$$I_L = \iiint_D \left[(x-1)^2 + (y-2)^2\right] \delta(x, y, z)\ dV.$$

Change of Variables: In multiple integrals, sometimes making a change of variables makes things a lot easier.

- Let $\iiint_R f(x,y,z)\ dx\ dy\ dz$ be a triple integral of a function over a region R, where R is a solid in the xyz space. We make a change of variable via

$$\begin{aligned} x &= g_1(u,v,w) \\ y &= g_2(u,v,w) \\ z &= g_3(u,v,w). \end{aligned}$$

  After making this change R becomes a region G in the uvw-space.

- The Jacobian of the change of variable is given by the determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

- Note that *the old variables are differentiated with respect to the new variables.*

- The integral $\iiint_R f(x,y,z)\ dx\ dy\ dz$ will be transformed as follows

$$\iiint_G f(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|\ dw\ dv\ du.$$

- Notice that the absolute value $|\cdot|$ of the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ is taken.

- Few important examples of change of variables
  (a) $R : x^2/a^2 + y^2/b^2 \le 1$ on xy coordinate system. Change of variables; $x = ar\cos\theta,\ y = br\sin\theta$. New region $G : r \le 1$ is the unit circle. Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)} = abr$.
  (b) $R : x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$ on xyz coordinate system. Change of variables; $x = a\rho\sin\phi\cos\theta,\ y = b\rho\sin\phi\sin\theta,\ z = c\rho\cos\phi$. New region $G : \rho \le 1$ is the unit sphere. Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,,\phi,\theta)} = abc\rho^2\sin\phi$.

2. Chapter 13

Here is the main framework

- Curve: $\vec{r}(t) = x(t)\ \hat{i} + y(t)\ \hat{j} + z(t)\ \hat{k}$.
- Velocity: $\vec{v}(t) = \frac{d\vec{r}}{dt} = x'(t)\ \hat{i} + y'(t)\ \hat{j} + z'(t)\ \hat{k}$.
- Speed: $|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.
- Unit tangent vector: $\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$.
- Arc length parameter: $s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\ d\tau$, where t_0 is the initial time and t is the final time.
- $\frac{ds}{dt} = |\vec{v}(t)|$.
- Arc length: $\int_a^b |\vec{v}(t)|\ dt$, where $t = a$ is the initial time and $t = b$ is the final time.
- Curvature: $\kappa = \frac{|d\vec{T}/dt|}{|\vec{v}(t)|}$.
- Unit normal vector: $\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$.

- Binormal vector: $\vec{B} = \vec{T} \times \vec{N}$.
- Acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$.
- Tangential and normal components of acceleration: $\vec{a} = a_T\vec{T} + a_N\vec{N}$, where the tangential component is $a_T = \frac{d|\vec{v}|}{dt}$ and the normal component is $a_N = \kappa|\vec{v}|^2 = \sqrt{|\vec{a}|^2 - a_T^2}$.

3. Chapter 16

There are mainly two types of line integrals, Line integral of a scalar field and line integral in a vector field.

(1) Line integral of a scalar function: This is used to compute the area bounded above a curve and below a surface (see figure 16.5). This can also be used to compute the length, mass, centroid, and moment of inertia of a wire. Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k};\ \ a \le t \le b$ be the equation of a space curve, and $f(x, y, z)$ be a scalar function then the line integral of $f(x, y, z)$ over the curve is given by

$$\int_a^b f(x(t), y(t), z(t))|\vec{v}(t)|\, dt. \tag{3}$$

If $\vec{r}(t)$ is a planer curve (eg. $\vec{r}(t) = y(t)\hat{j} + z(t)\hat{k}$ is a planer curve on the yz-plane), and f is a function defined on the same plane (eg. $f(y, z)$ is a function defined on the yz-plane), then (3) gives the area of the wall bounded in between the curve and the function f (similar to figure 16.5).

If $f(x, y, z) = \delta(x, y, z)$ is the density of the wire, the (3) gives the mass of the wire.

If $f(x, y, z) = (x^2 + y^2)\delta(x, y, z)$, where $\delta(x, y, z)$ is the density of the wire, then (3) gives the moment of inertia of the wire with respect to the z-axis etc. See the Table 16.1 for a complete set of formula and observe the similarity with the formula given in page 2.

(2) Line integral in a vector field: This is used to compute the work, flow, circulation, flux. Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k};\ \ a \le t \le b$ be a space curve, and $\vec{F}(x, y, z) = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$ be a vector field.

   (a) *Work:* The work done while moving a particle from the point $\vec{r}(a)$ to the point $\vec{r}(b)$ is given by

$$\int_a^b \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt}\, dt = \int_a^b (M\ dx + N\ dy + P\ dz). \tag{4}$$

   There is a list of equivalent formula given in the Table 16.2.

   (b) *Flow:* Flow of the vector field $\vec{F}$ along the curve $\vec{r}(t)$ from the point $\vec{r}(a)$ to the point $\vec{r}(b)$ is given by

$$\int_a^b \vec{F} \cdot d\vec{r}. \tag{5}$$

   Note that mathematically (4) and (5) are the same.

   If the curve starts and ends at the same point i.e., $\vec{r}(a) = \vec{r}(b)$, then the flow is called the *circulation* of the vector field along the curve.

   (c) *Flux:* Flux of a vector field $\vec{F}(x, y) = M(x, y)\ \hat{i} + N(x, y)\ \hat{j}$ across a simple closed curve $C \equiv \vec{r}(t) = x(t)\ \hat{i} + y(t)\ \hat{j}$ is given by

$$\oint_C (M\ dy - N\ dx). \tag{6}$$

   $\oint$ indicates that the curve C is a closed curve.

Conservative Vector Field:

- Let $f(x,y,z)$ be a scalar function, then the gradient vector field of f is denoted by $\vec{\nabla} f$ and defined as

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}. \tag{7}$$

- $\vec{\nabla} f$ gives the orthogonal vector to the surface defined by $f(x,y,z) = c$ for any constant c.
- A vector field $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ is conservative

$$\begin{aligned}
&\iff && \text{There exists a potential function } f \text{ such that } \vec{F} = \vec{\nabla} f.\\
&\iff && \vec{\nabla} \times \vec{F} = 0.\\
&\iff && \oint_C \vec{F}\cdot d\vec{r} = 0, \text{ for any simple closed curve } C.\\
&\iff && \int_A^B \vec{F}\cdot d\vec{r} \text{ is independent of the path which connects the two points } A \text{ and } B
\end{aligned} \tag{8}$$

[' $\iff$ ' indicates that the statements are equivalent, $\vec{\nabla} \times \vec{F}$ is defined in (19)*].*

- *Finding the Potential Function* Let $f(z,y,z)$ be the potential function of the conservative vector field $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$. Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= M(x,y,z) && \Rightarrow f(x,y,z) = \int M(x,y,z)\,dx + C_1(y,z)\\
\frac{\partial f}{\partial y} &= N(x,y,z) && \Rightarrow f(x,y,z) = \int N(x,y,z)\,dy + C_2(x,z)\\
\frac{\partial f}{\partial z} &= P(x,y,z) && \Rightarrow f(x,y,z) = \int P(x,y,z)\,dz + C_3(x,y).
\end{aligned}$$

To find the $f(x,y,z)$, add the above three equations *but ignore the repeated expressions.* For example, if

$$\int M(x,y,z)\,dx = e^x \cos y + xyz \tag{9}$$

$$\int N(x,y,z)\,dy = xyz + e^x \cos y \tag{10}$$

$$\int P(x,y,z)\,dz = xyz + \frac{z^2}{2}, \tag{11}$$

then

$$f(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C. \tag{12}$$

Note that $e^x \cos y$ was present in the equations (9) and (10) but it was written only once in (12). Similarly, xyz was present in all of (9), (10), and (11), but it was written only once in (12).

Alternatively, you may use the method described in the book.

- If a particle is moving from the point A to the point B in a conservative force field, then the work done is independent of the path and by the fundamental theorem of line integral, (4) and (8) become

$$\text{Work} = \int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A),$$

where f is the potential function of $\vec{F}$.

Green's theorem on plane: Let $\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$ be a vector field.

- We can always use the formula (4) to find the work done or flow and (6) to find the flux (in xy-plane) in any vector field, regardless of whether it is conservative or not.

- *Green's theorem (circulation):* If M and N are differentiable functions of x, y, and C is a *simple closed curve oriented counter-clockwise*, then (4) can be written as

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C (M\ dx + N\ dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA, \tag{13}$$

where S is the region enclosed by the simple closed curve C. dA can be taken as $dy\ dx$, $dx\ dy$, or $r\ dr\ d\theta$ as per convenience.

- We know that if the vector field $\vec{F}$ is conservative, then work done or flow along a closed curve is zero. Because if $\vec{F}$ is conservative, then $\vec{\nabla} \times \vec{F} = 0$. Which implies that $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$. As a result, from (13), we have $\oint_C \vec{F} \cdot d\vec{r} = 0$.

- *Green's theorem (flux):* Let C be a simple closed curve in the xy-plane and S be the region enclosed by C, then (6) can be written as

$$\oint_C (M\ dy - N\ dx) = \iint_S \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA, \tag{14}$$

where dA can be taken as $dx\ dy$, $dy\ dx$, or $r\ dr\ d\theta$.

- Note that the theorem is also valid in xz and yz planes. For example, if $\vec{F} = M(x,z)\,\hat{i} + N(x,z)\,\hat{j}$ and C is a simple closed curve in xz-plane, then

$$\oint_C (M\ dx + N\ dz) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial z} \right) dA,$$

where S is the region enclosed by the curve C and $dA = dx\ dz = dz\ dx = r\ dr\ d\theta$.

- *circulation density and flux density:* Let $\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$ be a vector field.

$$\text{Circulation density at } (x,y) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$
$$\text{Flux density at } (x,y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

- Note that (13) and (14) are mathematically same formula. For example, we can derive (14) from (13)

$$\begin{aligned}
\oint_C (M\ dy - N\ dx) &= \oint_C [(-N)\ dx + M\ dy] \\
&= \iint_S \left[\frac{\partial M}{\partial x} - \left(-\frac{\partial N}{\partial y} \right) \right] dA \quad \text{(using (13))} \\
&= \iint_S \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA.
\end{aligned}$$

Similarly, (13) can be derived from (14).

- *Area of the region enclosed by a simple closed curve:* Let S be the region enclosed by the simple closed curve C. Then applying the Green's theorem we have

$$\begin{aligned}\frac{1}{2}\oint_C (x\,dy - y\,dx) &= \frac{1}{2}\iint_S \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right)\,dA \\ &= \iint_S dA \\ &= \text{Area of } S.\end{aligned}$$

So the formula $\frac{1}{2}\oint_C (x\,dy - y\,dx)$ can be used to find the area of S.

Surface Integrals: Like line integrals, there are basically two types of surface integrals; surface integral of a scalar function and surface integral in a vector field.

Surface integral of a scalar function: This is used to find the area, mass, centroid, moment of inertia of a three dimensional surface. The general framework is given below.

- Let S be a three dimensional surface. For example, $S: x^2 + y^2 + z^2 = a^2,\ z \geq 0$ is the surface of the upper hemisphere of radius a. Then surface integral of a function $G(x, y, z)$ is given by

$$\mathcal{I} = \iint_S G(x, y, z)\,d\sigma, \tag{15}$$

  where $d\sigma$ is the infinitesimal surface area of S (like $dx\,dy$ on xy-plane).

- If $G(x, y, z) = 1$, then (15) gives the surface area of S. If $G(x, y, z) = \delta(x, y, z)$ is the density of S, then (15) gives the mass of the surface S. If $G(x, y, z) = (x^2 + y^2)\delta(x, y, z)$, then (15) gives the moment of inertia of S with respect to the z-axis. See the table 16.3 for a complete set of formula and compare those with the formula listed on page 2 of this review.

- *finding $d\sigma$:* There are two methods to find $d\sigma$.

Projection method: In this method, we take the projection/shadow of the surface on to one of the coordinate planes. While taking the shadow, we need to keep in mind that the projection *(a) must not be a line/curve*, and *(b) there should not be any overlapping.* For example, consider the surface

$$S: x^2 + y^2 - 1 = 0,\ y \geq 0,\ 0 \leq z \leq 2. \tag{16}$$

This is half of a cylindrical surface which is standing on the curve $x^2 + y^2 = 1$ and 2 unit tall towards the positive z-axis. We must not take the shadow of this surface on the xy-plane. Because then we will only get the curve $x^2 + y^2 = 1$ which violets the first requirement. We may take the shadow on xz-plane (or yz-plane). But the shadows of $x = \pm\sqrt{1 - y^2}$ will overlap on yz-plane. So, we should take the shadow on the xz plane. The shadow on the xz-plane is the rectangle $-1 \leq x \leq 1,\ 0 \leq z \leq 2$.

Let $f(x, y, z) = 0$ be the equation of the surface S, and R be the shadow of the surface S on xy-plane. Then $d\sigma$ is given by

$$d\sigma = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \hat{k}|}\,dA,$$

where dA can be taken as $dx\,dy$, $dy\,dx$, or $r\,dr\,d\theta$ as per convenience. The limits of the integration are determined by the shadow R. For other coordinate planes

$$\begin{aligned} d\sigma &= \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \hat{j}|}\, dA, \quad (xz\text{-plane, } dA = dx\,dz = dz\,dx = r\,dr\,d\theta) \\ d\sigma &= \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \hat{i}|}\, dA, \quad (yz\text{-plane, } dA = dy\,dz = dz\,dy = r\,dr\,d\theta). \end{aligned}$$

In our example (16), $f(x, y, z) = x^2 + y^2 - 1$. Therefore $\vec{\nabla} f = 2x\,\hat{i} + 2y\,\hat{j}$. If we take the shadow on the xz-plane then $\vec{\nabla} f \cdot \hat{j} = 2y$, and the shadow is $R: \ -1 \le x \le 1,\ 0 \le z \le 2$. Then the equation (15) becomes

$$\begin{aligned} \mathcal{I} &= \iint_R G(x, \sqrt{1 - x^2}, z)\, \frac{2\sqrt{x^2 + y^2}}{|2y|}\, dA \\ &= \int_{-1}^{1} \int_0^2 G(x, \sqrt{1 - x^2}, z) \frac{1}{\sqrt{1 - x^2}}\, dz\, dx. \end{aligned}$$

Note that y is replaced by $\sqrt{1 - x^2}$ which is obtained from the equation of the surface (16).

Parametrization method: In this method, the equation of surface is given by two parameters. For example,

$$S: \ \vec{r}(u, v) = \cos v\,\hat{j} + \sin v\,\hat{i} + u\,\hat{k}, \quad 0 \le v \le \pi,\ 0 \le u \le 2, \tag{17}$$

is a parametrization of the same surface (16). Then $d\sigma$ is given by

$$d\sigma = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|.$$

The limits of the integration is determined by the limits of the parametrization. In the above example, $\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = 1$. Therefore the integration (15) will become

$$\mathcal{I} = \int_0^{\pi} \int_0^2 G(\cos v, \sin v, u)\, du\, dv.$$

Note that x, y, z in G are replaced by the parametrization (17) of S.

Surface integral in a vector field: This is used to find the flux of a vector field across a surface.

- Let S be a surface and $\vec{F}$ be a vector field. Then the flux of $\vec{F}$ towards the unit normal vector $\hat{n}$ to the surface is given by

$$\text{Flux} = \iint_S (\vec{F} \cdot \hat{n})\, d\sigma, \tag{18}$$

where $d\sigma$ can be found using the method described above.

- *Finding $\hat{n}$:* If the equation of the surface is given in the implicit form $f(x, y, z) = 0$, then the unit normal vector is given by

$$\hat{n} = \pm \frac{\vec{\nabla} f}{|\vec{\nabla} f|}.$$

If the equation of the surface is given in the parametric form $\vec{r}(u,v)$, then $\hat{n}$ is given by

$$\hat{n} = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|},$$

where the sign $\pm$ is chosen from the description of the problem.

Stokes' Theorem: Let $\vec{F} = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$ be a vector field.

- Curl of the vector field $\vec{F}$ is denoted by $\vec{\nabla} \times \vec{F}$ and defined as

$$\begin{aligned} \vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x,y,z) & N(x,y,z) & P(x,y,z) \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \hat{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{k}. \end{aligned} \quad (19)$$

- *Stokes' Theorem:* Let C be the (piecewise smooth) counter-clockwise oriented boundary of a surface S. Note that C is a simple closed curve. Flow of a vector field $\vec{F}$ along C which is given by (5) can be written as

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\; d\sigma, \quad (20)$$

  where $\hat{n}$ is the unit normal vector to the surface S and the direction of $\hat{n}$ is determined by the orientation of C using right-hand rule.

- Stokes' theorem is three dimensional analogue of Green's theorem of flow. More precisely, (13) can be obtained from (20).

Divergence Theorem: Let $\vec{F} = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$ be a vector field, S be a closed surface and D be the region enclosed by S.

- Divergence of $\vec{F}$ is denoted by $\vec{\nabla} \cdot \vec{F}$ and defined as

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \quad (21)$$

- Do not get confused between (7), (21), and (19).

- *Divergence Theorem:* The outward flux of the vector field $\vec{F}$ across the closed surface S which is defined by (18) can be written as

$$\oiint_S (\vec{F} \cdot \hat{n})\; d\sigma = \iiint_D (\vec{\nabla} \cdot \vec{F})\; dV.$$

  Note that $\oiint$ indicates that the surface S is closed.