**MAT 21D** Max time 20mins **Quiz 7**

Name:________________________________________________ September 7, 2016

> All unnecessary electronics must be turned off and out of sight. This means no calculator, cellular phones, iPods, wearing of headphones, or anything of the sort. This is a closed book and closed notes test.

**Problem 1:** *(10 points)* [Divergence theorem] Let $D$ be the cube cut from the first octant by the planes $x = 1,\ y = 1$, and $z = 1$. Find the outward flux of the vector field $\vec{F} = x^2\ \hat{i} + y^2\ \hat{j} + z^2\ \hat{k}$ across the surface of the cube.

**Solution:** Let $S$ be the surface of the cube $D$. We notice that $S$ is a closed surface. Then the outward flux across the surface of the cube is $\oiint_S (\vec{F}\cdot\hat{n})\ d\sigma$. Using the Divergence theorem, we have

$$\oiint_S (\vec{F}\cdot\hat{n})\ d\sigma = \iiint_D (\vec{\nabla}\cdot\vec{F})\ dV.$$

We compute

$$\vec{\nabla}\cdot\vec{F} = 2x + 2y + 2z.$$

Therefore the outward flux of the vector field $\vec{F}$ across the surface of the cube is

$$\begin{aligned} \oiint_S (\vec{F}\cdot\hat{n})\ d\sigma &= \iiint_D (\vec{\nabla}\cdot\vec{F})\ dV \\ &= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\ dz\ dy\ dx \\ &= 3. \end{aligned}$$

**Problem 2:** *(10 points)* [Stokes' theorem] Suppose $\vec{F} = \vec{\nabla} \times \vec{A}$, where

$$\vec{A} = (y + \sqrt{z})\ \hat{i} + e^{xyz}\ \hat{j} + \cos(xz)\ \hat{k}.$$

Determine the flux of $\vec{F}$ outward through the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

*[Hint: Boundary of the hemisphere is the circle $x^2 + y^2 = 1,\ z = 0$.]*

**Solution:** Let us denote the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ by $S$. From the definition of flux, the outward flux of $\vec{F}$ through the surface of $S$ is $\iint_S (\vec{F} \cdot \hat{n})\ d\sigma$, where $\hat{n}$ is the outward unit normal vector to $S$. But it is given that $\vec{F} = \vec{\nabla} \times \vec{A}$. Therefore using the Stokes' theorem, we have

$$\begin{aligned}
\iint_S (\vec{F} \cdot \hat{n})\ d\sigma &= \iint_S \left[(\vec{\nabla} \times \vec{A}) \cdot \hat{n}\right]\ d\sigma \\
&= \oint_C \vec{A} \cdot d\vec{r}, \quad \text{(Stokes' theorem)}
\end{aligned}$$

where $C$ is the boundary of the hemisphere $S$.

We observe that the boundary of the hemisphere $S$ is the unit circle $x^2 + y^2 = 1, z = 0$ on the $xy$-plane, which can be parametrized as

$$\vec{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j},\ 0 \leq t \leq 2\pi.$$

Note that the above circle is oriented counter-clockwise. Since we are finding the outward flux through the hemisphere, the unit normal vector $\hat{n}$ is pointed towards the outward direction, which matches with the counter-clockwise rotation of a screw on the surface. Therefore

$$\begin{aligned}
\oint_C \vec{A} \cdot d\vec{r} &= \int_0^{2\pi} \left(\sin t\ \hat{i} + \hat{j} + \hat{k}\right) \cdot (-\sin t\ \hat{i} + \cos t\ \hat{j})\ dt \\
&= \int_0^{2\pi} (-\sin^2 t + \cos t)\ dt \\
&= \int_0^{2\pi} \left[-\frac{1}{2}(1 - \cos 2t) + \cos t\right]\ dt \\
&= -\pi.
\end{aligned}$$