MAT 21D Max time 20mins **Quiz 5**

Name:____ August 24, 2016

All unnecessary electronics must be turned off and out of sight. This means no calculator, cellular phones, iPods, wearing of headphones, or anything of the sort. This is a closed book and closed notes test.

Problem 1: *(7 points)* Find the mass of the wire lying along the infinite curve $\vec{r}(t) = t\,\hat{i} + t\,\hat{j} + t\,\hat{k},\ \ 1 \le t < \infty$, whose density is given by the function $f(x, y, z) = \sqrt{3}/(x^2 + y^2 + z^2)$.

Solution: From the equation of the curve, we can compute

$$\vec{v}(t) = \hat{i} + \hat{j} + \hat{k} \qquad \textit{(2 points)}.$$

Mass of the wire is given by

$$\begin{aligned} M &= \int_1^\infty f(x(t), y(t), z(t))|\vec{v}(t)|\,dt \qquad \textit{(2 points)} \\ &= \int_1^\infty \frac{\sqrt{3}}{t^2 + t^2 + t^2}\sqrt{3}\,dt \\ &= \int_1^\infty \frac{1}{t^2}\,dt \\ &= -\frac{1}{t}\bigg|_1^\infty = 1. \qquad \textit{(3 points)} \end{aligned}$$

Problem 2:*(13 points)* Consider the vector field

$$\vec{F} = 3x^2\,\hat{\imath} + \frac{z^2}{y}\,\hat{\jmath} + 2z\ln y\,\hat{k}.$$

(a) *(5 points)* Prove that $\vec{F}$ is a conservative vector field.

(b) *(5 points)* Find the potential function.

(c) *(3 points)* Find the work done by moving a particle from $(1,1,1)$ to $(1,2,3)$.

Solution: (a) The curl of the vector field is

$$\begin{aligned}
\vec{\nabla}\times\vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & \frac{z^2}{y} & 2z\ln y \end{vmatrix} \qquad \textit{(2 points)} \\
&= \left[\frac{\partial}{\partial y}(2z\ln y) - \frac{\partial}{\partial z}(z^2/y)\right]\hat{\imath} + \left[\frac{\partial}{\partial y}(3x^2) - \frac{\partial}{\partial x}(2z\ln y)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(z^2/y) - \frac{\partial}{\partial y}(3x^2)\right]\hat{k} \\
&= \left(\frac{2z}{y} - \frac{2z}{y}\right)\hat{\imath} + (0-0)\,\hat{\jmath} + (0-0)\,\hat{k} \qquad \textit{(2 points)} \\
&= \vec{0} \qquad \textit{(1 point).}
\end{aligned}$$

Therefore $\vec{F}$ is a conservative vector field.

(b) Let $f(z,y,z)$ be the potential function. Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3x^2 & &\Rightarrow f(x,y,z) = x^3 + C_1(y,z) \\
\frac{\partial f}{\partial y} &= \frac{z^2}{y} & &\Rightarrow f(x,y,z) = z^2\ln y + C_2(x,z) \qquad \textit{(3 points)} \\
\frac{\partial f}{\partial z} &= 2z\ln y & &\Rightarrow f(x,y,z) = z^2\ln y + C_3(x,y).
\end{aligned}$$

Therefore the potential function is given by $f(x,y,z) = x^3 + z^2\ln y + C$. *(2 points)*

(c) The work done by moving a particle from $(1,1,1)$ to $(1,2,3)$ is given by

$$\begin{aligned}
& f(1,2,3) - f(1,1,1) \qquad \textit{(2 points)} \\
=\; & (1 + 9\ln 2) - (1 + \ln 1) = 9\ln 2. \qquad \textit{(1 point)}
\end{aligned}$$