MAT 21D Max time 20mins **Quiz 4**

Name:____ August 18, 2016

All unnecessary electronics must be turned off and out of sight. This means no calculator, cellular phones, iPods, wearing of headphones, or anything of the sort. This is a closed book and closed notes test.

Problem 1: *(12 points)* Find the unit tangent vector $\vec{T}$ and the curvature κ of the following curve

$$\vec{r}(t) = (\cos^3 t)\hat{i} + (\sin^3 t)\hat{j}.$$

Solution: We compute

$$\begin{aligned}
\vec{v}(t) = \frac{d\vec{r}}{dt} &= (-3\cos^2 t \sin t)\hat{i} + (3\sin^2 t \cos t)\hat{j} \qquad \textit{(2 points)} \\
&= 3\cos t \sin t(-\cos t\ \hat{i} + \sin t\ \hat{j}) \\
\text{Therefore, } |\vec{v}(t)| &= 3|\cos t \sin t|\sqrt{(-\cos t)^2 + \sin^2 t} = 3|\cos t \sin t| \qquad \textit{(2 points)} \\
\text{So, } \vec{T} &= \frac{\vec{v}(t)}{|\vec{v}(t)|} = \begin{cases} -\cos t\ \hat{i} + \sin t\ \hat{j} & \text{if } \cos t \sin t > 0 \\ \cos t\ \hat{i} - \sin t\ \hat{j} & \text{if } \cos t \sin t < 0 \end{cases} \qquad \textit{(2 points)} \\
\text{Consequently, } \frac{d\vec{T}}{dt} &= \begin{cases} \sin t\ \hat{i} + \cos t\ \hat{j} & \text{if } \cos t \sin t > 0 \\ -\sin t\ \hat{i} - \cos t\ \hat{j} & \text{if } \cos t \sin t < 0. \end{cases} \qquad \textit{(2 points)}
\end{aligned}$$

So, the curvature is

$$\begin{aligned}
\kappa &= \frac{1}{|\vec{v}(t)|}\left|\frac{d\vec{T}}{dt}\right| \qquad \textit{(2 points)} \\
&= \frac{1}{3|\cos t \sin t|}\sqrt{\sin^2 t + \cos^2 t} \\
&= \frac{1}{3|\cos t \sin t|}. \qquad \textit{(2 points)}
\end{aligned}$$

- Take off *1 point* if it is written as $3\cos t \sin t$ instead of $|3\cos t \sin t|$ in the expression of $|\vec{v}(t)|$.
- Take off points if the formula of κ is written wrong (eg. missing $|\cdot|$ etc.).
- Take off *1 or 2 points* if the *arrow* sign is missing from some of the vector quantities.

Problem 2: *(8 points)* Find the length of the curve

$$\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}, \quad \frac{\pi}{2} \le t \le \pi.$$

Solution: Differentiating $\vec{r}(t)$ with respect to t, we obtain

$$\begin{aligned} \vec{v}(t) &= (-\sin t + \sin t + t\cos t)\,\hat{i} + (\cos t - \cos t + t\sin t)\,\hat{j} \\ &= t(\cos t\,\hat{i} + \sin t\,\hat{j}). \qquad \textit{(2 points)} \end{aligned}$$

Therefore

$$|\vec{v}(t)| = |t|. \qquad \textit{(2 points)}$$

Using the formula for arc length, we can calculate the length of the curve as follows

$$\begin{aligned} L &= \int_{\pi/2}^{\pi} |\vec{v}(t)|\,dt \qquad \textit{(2 points)} \\ &= \int_{\pi/2}^{\pi} t\,dt \\ &= \frac{1}{2}\left[\pi^2 - \frac{\pi^2}{4}\right] = \frac{3\pi^2}{8}. \qquad \textit{(2 points)} \end{aligned}$$