MAT 21D Max time 20mins **Quiz 3**

Name:____ August 15, 2016

All unnecessary electronics must be turned off and out of sight. This means no calculator, cellular phones, iPods, wearing of headphones, or anything of the sort. This is a closed book and closed notes test.

Problem 1: *(6 points)* Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ of the transformation

$$x = u\cos v, \quad y = u\sin v, \quad z = w.$$

Solution: The Jacobian is given by

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad \textit{(2 points)} \\
&= \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \textit{(2 points)} \\
&= \cos v \begin{vmatrix} u\cos v & 0 \\ 0 & 1 \end{vmatrix} - (-u\sin v) \begin{vmatrix} \sin v & 0 \\ 0 & 1 \end{vmatrix} + 0 \\
&= u\cos^2 v + u\sin^2 v = u \quad \textit{(2 points)}.
\end{aligned}$$

- If the first line is missing but still the second line is correct: *+4 points* and *+2 points* if the final is correct.
- If the Jacobian is transposed but everything else are correct: *Full credit.*
- If the first line is missing and the second line is incorrect: *max 4 points.*

Problem 2: *(14 points)* Find the mass and the centroid of the solid region which is bounded on the top by the paraboloid $z = x^2 + y^2$, on the bottom by the plane $z = 0$, and on the sides by the cylinder $x^2 + y^2 = 1$. The density of the solid is $\delta(x, y, z) = z$.

[Hint: Notice the symmetry.]

Solution: Using the cylindrical coordinate system, the mass of the solid is

$$\begin{aligned}
M &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} \delta(x, y, z) \; dz \; r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 \int_0^{r^2} z \; dz \; r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 \frac{1}{2} z^2 \Big|_0^{r^2} \; r dr d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^5 \; dr d\theta \\
&= \frac{1}{2} \times \frac{1}{6} \times 2\pi = \frac{\pi}{6}.
\end{aligned}$$

Since the solid is symmetric with respect to the z-axis and <u>*the density is also symmetric with respect to the z-axis*</u>, the centroid must lie on the z-axis i.e., $\bar{x} = 0 = \bar{y}$. To find the $\bar{z}$, we calculate the first moment with respect to the xy plane

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} z\delta(x, y, z) \; dz \; r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 \int_0^{r^2} z^2 \; dz \; r dr d\theta \\
&= \frac{1}{3} \int_0^{2\pi} \int_0^1 r^7 \; dr d\theta \\
&= \frac{1}{3} \times \frac{1}{8} \times 2\pi = \frac{\pi}{12}.
\end{aligned}$$

Therefore the z coordinate of the centroid is $\bar{z} = \frac{M_{xy}}{M} = \frac{1}{2}$. So the centroid is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 1/2)$.

- Picture: *2 points.*
- Mass: *4 points.*
- Symmetry Statement: Solid is symmetric w.r.to z-axis *(1 point)* + Density is symmetric w.r.to z-axis *(1 point)*.
- M_{xy}: *4 points.*
- Final answer: *2 points.*

(*) If the method and the answer is correct without the picture: *Full credit.*