MAT 21D Max time: 1 hr 20 mins **Practice Midterm 2 Solutions**

Name:____ August 24, 2016

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).
- You may bring hand written notes ONLY ON ONE SIDE of a half page (where full page = max A4).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- Check that you have all **seven** pages of the exam. (The number of pages includes this cover page.)

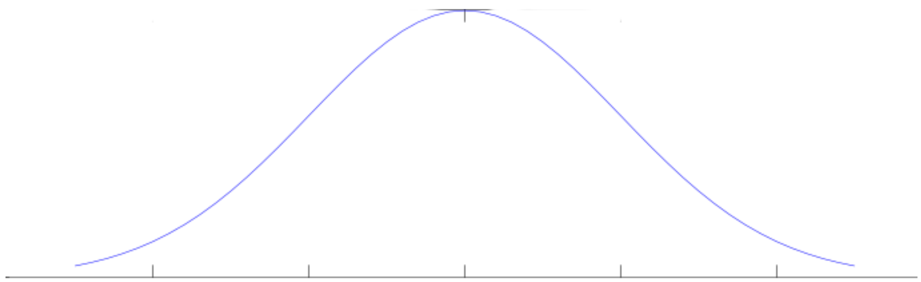
During the exam:

- Keep your eyes on your own exam!

Note that the exam length is exactly 1 hr 20 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	5	6	**Total**
Score							

Problem 1: *(10 points)* Find the area of the region bounded between the graph of the function $f(x) = e^{-x^2}$ and the x axis.



Solution: Clearly from the picture, the area is given by $A = \int_{-\infty}^{\infty} e^{-x^2}\ dx$. Then we can also write $A = \int_{-\infty}^{\infty} e^{-y^2}\ dy$.

$$\begin{aligned}
A^2 &= A \times A \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right) \times \left(\int_{-\infty}^{\infty} e^{-y^2}\ dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\ rdrd\theta \quad \text{(transforming into polar coordinate)} \\
&= \int_{0}^{2\pi} \left[-\frac{1}{2}e^{-r^2}\Big|_0^{\infty} \right]\ d\theta \\
&= \frac{1}{2}\int_{0}^{2\pi} d\theta = \pi.
\end{aligned}$$

Therefore $A = \sqrt{\pi}$.

Problem 2: *(10 points)* Draw the region of the integration and evaluate the following integral

$$\int_0^1 \int_{2y}^2 4\cos(x^2)\, dxdy$$

Solution:

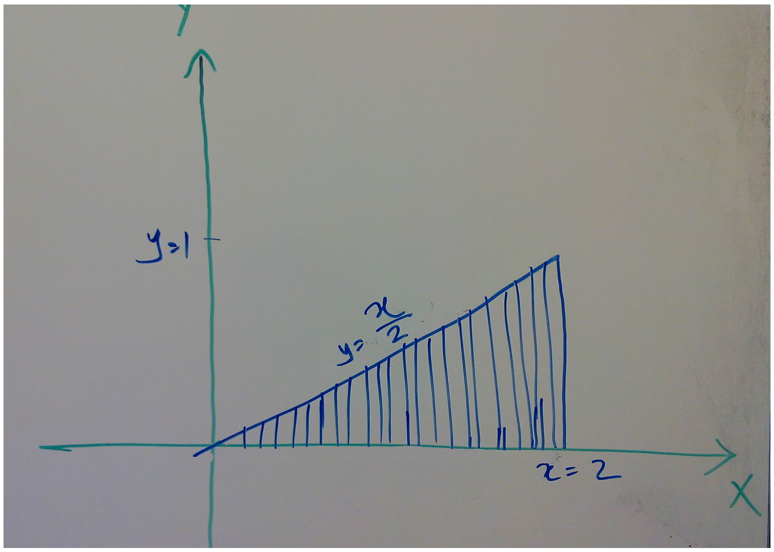


Figure 1: Region of integration

From the above picture, reversing the order of integration we obtain

$$\begin{aligned}
\int_0^1 \int_{2y}^2 4\cos(x^2)\, dxdy &= \int_0^2 \int_0^{x/2} 4\cos(x^2)\, dydx \\
&= \int_0^2 2x\cos(x^2)\, dx \\
&= \sin(x^2)\Big|_0^2 \\
&= \sin 4.
\end{aligned}$$

Problem 3: *(15 points)* Find the mass of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where the density is given by $\delta(x,y,z)=\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}$. *[Hint: Let x=au, y=bv, and z=cw. Then find the mass of an appropriate region in uvw-space.]*

Solution: Mass of the ellipsoid is given by

$$\begin{aligned} M &= \iiint_{ellipsoid} \delta \, dV \\ &= \iiint_{ellipsoid} \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}} \, dzdydx. \end{aligned}$$

Let us make the substitution $x=au, y=bv, z=cw$. Then in the uvw-space, the equation of the ellipsoid becomes $u^2+v^2+w^2=1$, which is a unit sphere. And the density function becomes $\delta(u,v,w)=\sqrt{u^2+v^2+w^2}$. Jacobian of this transformation is

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \\ &= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc. \end{aligned}$$

Therefore the mass of the ellipsoid is

$$\begin{aligned} M &= |abc| \iiint_{sphere} \sqrt{u^2+v^2+w^2} \, dwdvdu \\ &= |abc| \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho\cdot\rho^2 \sin\phi \, d\phi d\theta \quad \text{(changing it into spherical coordinates)} \\ &= \frac{|abc|}{4}\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi d\theta \\ &= \frac{|abc|}{4}\int_0^{2\pi} [-\cos\phi|_0^{\pi}] \, d\theta \\ &= \frac{|abc|}{2}\int_0^{2\pi} d\theta \\ &= \pi|abc|. \end{aligned}$$

Problem 4: *(20 points)* Consider the following space curve

$$\vec{r}(t) = (e^t \cos t)\hat{\imath} + (e^t \sin t)\hat{\jmath} + 2\hat{k}.$$

(a) *(5 points)* Find the length of the curve from the point $(1,0,2)$ to the point $(0, e^{\pi/2}, 2)$.
(b) *(15 points)* Find the tangent vector $\vec{T}$, unit normal vector $\vec{N}$ and the curvature κ.

Solution: From the given equation of the curve we have

$$\begin{aligned}
\vec{v}(t) &= = \frac{d\vec{r}}{dt} = e^t(\cos t - \sin t)\hat{\imath} + e^t(\sin t + \cos t)\hat{\jmath} \\
|\vec{v}(t)| &= e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = e^t\sqrt{2}.
\end{aligned}$$

(a) The $\vec{r}(t)$ passes through the points $(1,0,2)$ and $(0, e^{\pi/2}, 2)$ when $t = 0$ and $t = \pi/2$ respectively. Therefore length of the curve between $(1,0,2)$ and $(0, e^{\pi/2}, 2)$ is

$$\begin{aligned}
L &= \int_0^{\pi/2} |\vec{v}(t)|\, dt \\
&= \int_0^{\pi/2} e^t\sqrt{2}\, dt \\
&= \sqrt{2}(e^{\pi/2} - 1).
\end{aligned}$$

(b) The tangent vector $\vec{T}$ is given by

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{\sqrt{2}}(\cos t - \sin t)\hat{\imath} + \frac{1}{\sqrt{2}}(\sin t + \cos t)\hat{\jmath}.$$

To find the unit normal vector $\vec{N}$ and the curvature κ, we need to compute

$$\begin{aligned}
\frac{d\vec{T}}{dt} &= -\frac{1}{\sqrt{2}}(\sin t + \cos t)\hat{\imath} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\hat{\jmath} \\
\left|\frac{d\vec{T}}{dt}\right| &= \frac{1}{\sqrt{2}}\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = 1.
\end{aligned}$$

Therefore the unit normal vector $\vec{N}$ and the curvature κ are given by

$$\begin{aligned}
\vec{N} &= \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = -\frac{1}{\sqrt{2}}(\sin t + \cos t)\hat{\imath} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\hat{\jmath}, \\
\kappa &= \frac{1}{|\vec{v}(t)|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{e^t\sqrt{2}}.
\end{aligned}$$

Problem 5: *(10 points)* Consider the vector field $\vec{F} = 2x\hat{\imath} - 3y\hat{\jmath}$, and the circle $\vec{r}(t) = (a\cos t)\hat{\imath} + (a\sin t)\hat{\jmath}$, $0 \le t \le 2\pi$. Find the circulation of the field along the circle, and the flux of the field across the circle.

Solution:

$$\begin{aligned}
\text{Circulation} &= \oint (M\ dx + N\ dy) \\
&= \int_0^{2\pi} [(2a\cos t)(-a\sin t) + (-3a\sin t)(a\cos t)]\ dt \\
&= -5a^2 \int_0^{2\pi} \sin t\cos t\ dt \\
&= -\frac{5a^2}{2}\int_0^{2\pi} \sin 2t\ dt = 0. \\
\text{Flux} &= \oint (M\ dy - N\ dx) \\
&= \int_0^{2\pi} [(2a\cos t)(a\cos t) - (-3a\sin t)(-\sin t)]\ dt \\
&= a^2\int_0^{2\pi} [2\cos^2 t - 3\sin^2 t]\ dt \\
&= a^2\int_0^{2\pi} [2(\cos^2 t - \sin^2 t) - \sin^2 t]\ dt \\
&= a^2\int_0^{2\pi} [2\cos 2t - 1/2 + (\cos 2t)/2]\ dt \\
&= a^2\int_0^{2\pi} \left[\frac{5}{2}\cos 2t - \frac{1}{2}\right]\ dt \\
&= -\pi a^2.
\end{aligned}$$

Alternative Method (Using Green's Theorem)

$$\begin{aligned}
\text{Circulation} &= \oint (M\ dx + N\ dy) \\
&= \int\int_{circle} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dxdy \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\text{Flux} &= \oint (M\ dy - N\ dx) \\
&= \int\int_{circle} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\ dxdy \\
&= \int\int_{circle} (2-3)\ dxdy \\
&= -\text{Area of the circle} = -\pi a^2.
\end{aligned}$$

Problem 6: *(15 points)* Consider a thick spherical shell whose inner radius is a and outer radius is b and the density is $\delta = 1$. Find the moment of inertia of this spherical with respect to a diameter.

Solution:

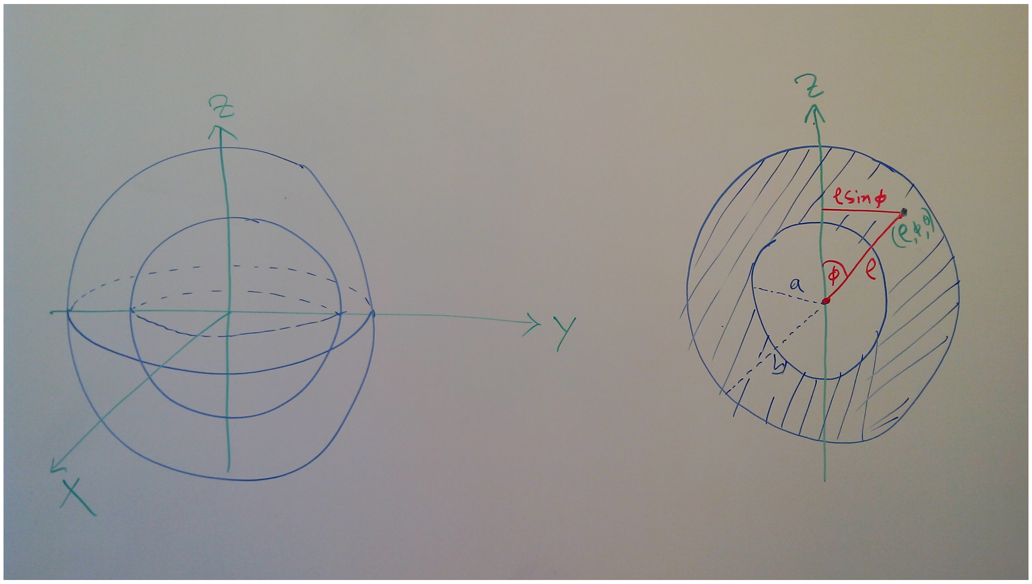


Figure 2: Left: The spherical shell/ Right: A vertical cross section of the shell

Since our object and it's density is spherically symmetric, the moment of inertia with respect to any diameter will be the same. Let us find the moment of inertia with respect to the $z-axis$. The moment of inertia with respect to the $z-axis$ is given by

$$\begin{aligned}
I_z &= \int\int\int (x^2+y^2)\delta\, dV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_a^b (\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)\cdot 1\cdot\ \rho^2\sin\phi\, d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_a^b \rho^4\sin^3\phi\, d\rho d\phi d\theta \\
&= \frac{1}{5}(b^5-a^5)\int_0^{2\pi}\int_0^{\pi}(1-\cos^2\phi)\sin\phi\, d\phi d\theta \\
&= \frac{1}{5}(b^5-a^5)\int_0^{2\pi}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\Big|_0^{\pi}\right] d\theta \\
&= \frac{1}{5}(b^5-a^5)\int_0^{2\pi}\frac{4}{3}\, d\theta \\
&= \frac{8\pi}{15}(b^5-a^5).
\end{aligned}$$