MAT 21D Max time: 1 hr 20 mins **Practice Midterm 1 Solutions**

Name:____ August 24, 2016

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).
- You may bring hand written notes ONLY ON ONE SIDE of a half page (where full page = max A4).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- Check that you have all **seven** pages of the exam. (The number of pages includes this cover page.)

During the exam:

- Keep your eyes on your own exam!

Note that the exam length is exactly 1 hr 20 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	5	6	**Total**
Score							

Problem 1:*(10 points)* Evaluate the following integral

$$\int_0^\infty e^{-x^2}\ dx.$$

Solution: Let $I = \int_0^\infty e^{-x^2}\ dx$. Then we can also write $I = \int_0^\infty e^{-y^2}\ dy$.

$$\begin{aligned}
I^2 &= I \times I \\
&= \left(\int_0^\infty e^{-x^2}\ dx\right) \times \left(\int_0^\infty e^{-y^2}\ dy\right) \\
&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\ dxdy \\
&= \int_0^{\pi/2} \int_0^\infty e^{-r^2}\ rdrd\theta \quad \text{(transforming into polar coordinate)} \\
&= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \Big|_0^\infty \right]\ d\theta \\
&= \frac{1}{2} \int_0^{\pi/2} d\theta = \pi/4.
\end{aligned}$$

Therefore $I = \sqrt{\pi}/2$.

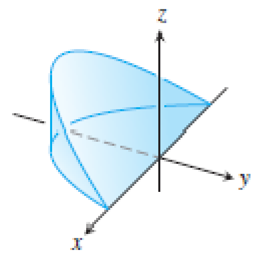
Problem 2: *(10 points)* Sketch the region of integration, and then evaluate the following integral

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\ dydx$$

Solution: Using the picture and reversing the order of integration we have

$$\begin{aligned}
\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\ dydx &= \int_0^1 \int_0^{3y^2} e^{y^3}\ dxdy \\
&= \int_0^1 3y^2 e^{y^3}\ dx \\
&= e^{y^3}\Big|_0^1 \\
&= e-1.
\end{aligned}$$

Problem 3: *(15 points)* Find the *y-coordinate* of the centroid of the wedge cut from the cylinder $x^2+y^2=1$ by the planes $z=-y$ and $z=0$, where the density is given by $\delta(x,y,z)=1$.



Solution: Mass of the solid is

$$\begin{aligned} M &= \int_{\pi}^{2\pi}\int_0^1\int_0^{-r\sin\theta} 1\cdot dz r dr d\theta \\ &= -\int_{\pi}^{2\pi}\int_0^1 r^2\sin\theta\, dr d\theta \\ &= -\frac{1}{3}\int_{\pi}^{2\pi}\sin\theta\, d\theta \\ &= \frac{1}{3}[\cos 2\pi-\cos\pi]=\frac{2}{3}. \end{aligned}$$

The first moment with respect to the xz-plane is

$$\begin{aligned} M_{xz} &= \int_{\pi}^{2\pi}\int_0^1\int_0^{-r\sin\theta} r\sin\theta\cdot dz r dr d\theta \\ &= -\int_{\pi}^{2\pi}\int_0^1 r^3\sin^2\theta\, d\theta \\ &= -\frac{1}{4}\int_{\pi}^{2\pi}\sin^2\theta\, d\theta \\ &= \frac{1}{8}\int_{\pi}^{2\pi}(\cos 2\theta-1)\, d\theta \\ &= -\pi/8. \end{aligned}$$

Therefore the y coordinate of the centroid is $M_{xz}/M=-3\pi/16$.

Problem 4:*(15 points)* Consider the following space curve

$$\vec{r}(t) = (e^t \cos t)\hat{\imath} + (e^t \sin t)\hat{\jmath} + 2\hat{k}.$$

(a) Find the length of the curve from the point $(1, 0, 2)$ to the point $(0, e^{\pi/2}, 2)$.
(b) Find the tangent vector $\vec{T}$, unit normal vector $\vec{N}$ and the curvature κ.

Solution: From the given equation of the curve we have

$$\begin{aligned} \vec{v}(t) &= = \frac{d\vec{r}}{dt} = e^t(\cos t - \sin t)\hat{\imath} + e^t(\sin t + \cos t)\hat{\jmath} \\ |\vec{v}(t)| &= e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = e^t\sqrt{2}. \end{aligned}$$

(a) The $\vec{r}(t)$ passes through the points $(1, 0, 2)$ and $(0, e^{\pi/2}, 2)$ when $t = 0$ and $t = \pi/2$ respectively. Therefore length of the curve between $(1, 0, 2)$ and $(0, e^{\pi/2}, 2)$ is

$$\begin{aligned} L &= \int_0^{\pi/2} |\vec{v}(t)|\, dt \\ &= \int_0^{\pi/2} e^t\sqrt{2}\, dt \\ &= \sqrt{2}(e^{\pi/2} - 1). \end{aligned}$$

(b) The tangent vector $\vec{T}$ is given by

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{\sqrt{2}}(\cos t - \sin t)\hat{\imath} + \frac{1}{\sqrt{2}}(\sin t + \cos t)\hat{\jmath}.$$

To find the unit normal vector $\vec{N}$ and the curvature κ, we need to compute

$$\begin{aligned} \frac{d\vec{T}}{dt} &= -\frac{1}{\sqrt{2}}(\sin t + \cos t)\hat{\imath} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\hat{\jmath} \\ \left|\frac{d\vec{T}}{dt}\right| &= \frac{1}{\sqrt{2}}\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = 1. \end{aligned}$$

Therefore the unit normal vector $\vec{N}$ and the curvature κ are given by

$$\begin{aligned} \vec{N} &= \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = -\frac{1}{\sqrt{2}}(\sin t + \cos t)\hat{\imath} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\hat{\jmath}, \\ \kappa &= \frac{1}{|\vec{v}(t)|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{e^t\sqrt{2}}. \end{aligned}$$

Problem 5: *(15 points)* Find the center of mass of a thin wire lying along the curve $\vec{r}(t) = t\hat{i}+2t\hat{j}+(2/3)t^{3/2}\hat{k}$, $0 \le t \le 2$, if the density is $\delta = 3\sqrt{5+t}$.

Solution: From the equation of the curve

$$\begin{aligned} \vec{v}(t) &= \frac{d\vec{r}}{dt} = \hat{i} + 2\hat{j} + \sqrt{t}\hat{k} \\ ds &= |\vec{v}(t)|\ dt = \sqrt{5+t}\ dt. \end{aligned}$$

Therefore the mass and the moments of the wire are given by

$$\begin{aligned} M &= \int_{t=0}^{2} \delta\ ds \\ &= 3\int_0^2 (5+t)\ dt \\ &= 3\left[5t + t^2/2\Big|_0^2\right] \\ &= 36, \\ M_{yz} &= \int_{t=0}^{2} x\delta\ ds \\ &= \int_0^2 3t(5+t)\ dt \\ &= \left[15t^2/2 + t^3\Big|_0^2\right] \\ &= 38, \\ M_{xz} &= \int_{t=0}^{2} y\delta\ ds \\ &= 6\int_0^2 t(5+t)\ dt \\ &= 76, \\ M_{xy} &= \int_{t=0}^{2} z\delta\ ds \\ &= 2\int_0^2 t^{3/2}(5+t)\ dt \\ &= 2\left[2t^{5/2} + \frac{2}{7}t^{7/2}\Big|_0^2\right] \\ &= 4\left[2^{5/2} + \frac{1}{7}2^{7/2}\right] \\ &= \frac{36}{7}2^{5/2}. \end{aligned}$$

Therefore the center of mass is

$$\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right) = \left(\frac{19}{18}, \frac{19}{9}, \frac{4\sqrt{2}}{7}\right).$$

Problem 6:*(15 points)* Consider the vector field $\vec{F} = (y \sin z)\hat{\imath} + (x \sin z)\hat{\jmath} + (xy \cos z)\hat{k}$.
(a) Is it a conservative vector field?
(b) Find the work done while moving a particle from $(0, 0, 0)$ to $(1, 1, 1)$ in this vector field.

Solution:

(a) Curl of the given vector field is

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \sin z & x \sin z & xy \cos z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y} xy \cos z - \frac{\partial}{\partial z} x \sin z\right)\hat{\imath} + \left(\frac{\partial}{\partial z} y \sin z - \frac{\partial}{\partial x} xy \cos z\right)\hat{\jmath} + \left(\frac{\partial}{\partial x} x \sin z - \frac{\partial}{\partial y} y \sin z\right) \\
&= (x \cos z - x \cos z)\hat{\imath} + (y \cos z - y \cos z)\hat{\jmath} + (\sin z - \sin z)\hat{k} \\
&= \vec{0}.
\end{aligned}$$

Therefore the given vector field is a conservative vector field.

(b) Since the vector field $\vec{F}$ is a conservative vector field, $\vec{F} = \vec{\nabla} f$ for some scalar function f, and the work done is $f(1, 1, 1) - f(0, 0, 0)$. From $\vec{F}$ we see that

$$\frac{\partial f}{\partial x} = y \sin z, \quad \frac{\partial f}{\partial y} = x \sin z, \quad \frac{\partial f}{\partial z} = xy \cos z.$$

Integrating each equation we have

$$\begin{aligned}
f(x, y, z) &= xy \sin z + C_1(y, z) \\
f(x, y, z) &= xy \sin z + C_2(x, z) \\
f(x, y, z) &= xy \sin z + C_3(x, y).
\end{aligned}$$

Combining all the equations, we obtain $f(x, y, z) = xy \sin z + C$, where C is a constant independent of x, y, z. Therefore the work done is $f(1, 1, 1) - f(0, 0, 0) = \sin 1$.