MAT 21D Max time: 1 hr 20 mins **Practice Final 2 (Solutions)**

Name:____ September 10, 2015

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).
- You may bring hand written notes ONLY ON ONE SIDE of a half page (where full page = max A4).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- Check that you have all **eight** pages of the exam. (The number of pages includes this cover page.)
- There is an *extra credit problem* on the last page.

During the exam:

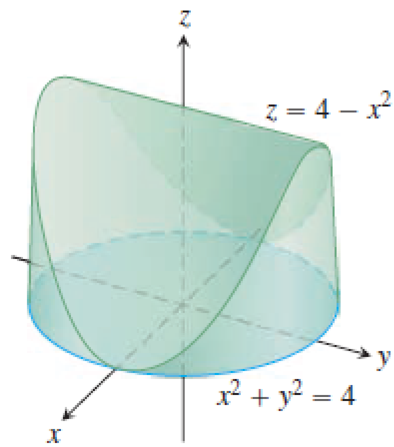
- Keep your eyes on your own exam!

Note that the exam length is exactly 1 hr 20 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	5	6	**Total**
Score							

Problem 1:*(10 points)* Find the mass of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by he xy-plane. The density of the solid is given by $\delta(x, y, z) = \sqrt{x^2 + y^2}$.

Solution:



Mass of the solid is

$$
\begin{aligned}
M &= \int\int_{x^2+y^2\le 4} \sqrt{x^2+y^2}(4-x^2)\, dydx \\
&= \int_0^{2\pi}\int_0^2 r(4-r^2\cos^2\theta)\, rdrd\theta \\
&= \int_0^{2\pi}\int_0^2 (4r^2 - r^4\cos^2\theta)\, drd\theta \\
&= \int_0^{2\pi} \frac{4}{3}r^3 - \frac{1}{5}r^5\cos^2\theta\Big|_0^2\, d\theta \\
&= \int_0^{2\pi}\left(\frac{32}{3} - \frac{16}{5} - \frac{16}{5}\cos 2\theta\right)\, d\theta \\
&= \left(\frac{32}{3} - \frac{16}{5}\right)2\pi \\
&= \frac{224\pi}{15}.
\end{aligned}
$$

Problem 2: *(10 points)* Find the moment of inertia with respect to a diameter of the solid sphere of radius a.

Solution: Since the sphere is a spherically symmetric object, the moment of inertia with respect to any diameter will be the same. Let us find the moment of inertia with respect to the $z-$axis. Density of the sphere is $\delta = 1$.

$$\begin{aligned}
I_z &= \int\int\int (x^2 + y^2)\delta \, dV \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} (\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta)\rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^4 \sin^3\phi \, d\rho d\phi d\theta \\
&= \frac{a^5}{5} \int_0^{2\pi} \int_0^{\pi} (1 - \cos^2\phi) \sin\phi \, d\phi d\theta \\
&= \frac{a^5}{5} \int_0^{2\pi} \left[-\cos\phi + \frac{1}{3}\cos^3\phi \Big|_0^{\pi} \right] d\theta \\
&= \frac{a^5}{5} \int_0^{2\pi} \frac{4}{3} \, d\theta \\
&= \frac{8\pi a^5}{15}.
\end{aligned}$$

Problem 3: *(15 points)* Consider the space curve

$$\vec{r}(t) = (\cos^3 t)\hat{\imath} + (\sin^3 t)\hat{\jmath}, \quad 0 \le t \le \pi/2.$$

(a) *(5 points)* Find the length of the curve.
(b) *(10 points)* Find the curvature κ.

Solution: (b) We compute

$$\begin{aligned}
\vec{v}(t) = \frac{d\vec{r}}{dt} &= (-3\cos^2 t \sin t)\hat{\imath} + (3\sin^2 t \cos t)\hat{\jmath} \\
&= 3\cos t \sin t(-\cos t\ \hat{\imath} + \sin t\ \hat{\jmath}) \\
\text{Therefore, } |\vec{v}(t)| &= 3|\cos t \sin t|\sqrt{(-\cos t)^2 + \sin^2 t} \\
&= 3|\cos t \sin t| \\
&= 3\cos t \sin t \quad (\text{since } 0 \le t \le \pi/2) \\
\text{So, } \vec{T} &= \frac{\vec{v}(t)}{|\vec{v}(t)|} \\
&= -\cos t\ \hat{\imath} + \sin t\ \hat{\jmath} \\
\text{Consequently, } \frac{d\vec{T}}{dt} &= \sin t\hat{\imath} + \cos t\hat{\jmath}.
\end{aligned}$$

So, the curvature is

$$\begin{aligned}
\kappa &= \frac{1}{|\vec{v}(t)|}\left|\frac{d\vec{T}}{dt}\right| \\
&= \frac{1}{3\cos t \sin t}\sqrt{\sin^2 t + \cos^2 t} \\
&= \frac{1}{3\cos t \sin t}.
\end{aligned}$$

(a) The length of the curve is

$$\begin{aligned}
L &= \int_0^{\pi/2} |\vec{v}(t)|\ dt \\
&= \int_0^{\pi/2} 3\cos t \sin t\ dt \\
&= \frac{3}{2}\int_0^{\pi/2} \sin 2t\ dt \\
&= -\frac{3}{4}\cos 2t\Big|_0^{\pi/2} \\
&= \frac{3}{2}.
\end{aligned}$$

Problem 4: *(10 points)* Consider the vector field

$$\vec{F} = (2x\ln y - yz)\hat{\imath} + \left(\frac{x^2}{y} - xz\right)\hat{\jmath} - xy\hat{k}.$$

Find the work done while moving a particle in the above vector field from $(1,2,1)$ to $(2,1,1)$ along the straight line.

Solution: Curl of the vector field $\vec{F}$ is given by

$$\begin{aligned} \vec{\nabla}\times\vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\ln y - yz & \frac{x^2}{y} - xz & -xy \end{vmatrix} \\ &= (-x+x)\,\hat{\imath} + (-y+y)\,\hat{\jmath} + \left(\frac{2x}{y} - z - \frac{2x}{y} + z\right)\hat{k} \\ &= \vec{0}. \end{aligned}$$

Therefore $\vec{F}$ is a conservative vector field. Let ϕ be the potential function of the vector field i.e., $\vec{F} = \vec{\nabla}\phi$. Then

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= 2x\ln y - yz \\ \frac{\partial\phi}{\partial y} &= \frac{x^2}{y} - xz \\ \frac{\partial\phi}{\partial z} &= -xy \end{aligned}$$

Integrating each differential equation with respect to the corresponding variables, we have

$$\begin{aligned} \phi(x,y,z) &= x^2\ln y - xyz + C_1(y,z) \\ \phi(x,y,z) &= x^2\ln y - xyz + C_2(x,z) \\ \phi(x,y,z) &= -xyz + C_3(x,y) \end{aligned}$$

Looking at the above equations, we see that if we take $C_1(y,z) = C = C_2(x,z)$, and $C_3(x,y) = x^2\ln y + C$, then the three equations of ϕ become he same. Therefore $\phi(x,y,z) = x^2\ln y - xyz + C$, where C is an arbitrary constant which is independent of x, y, z.

Since $\vec{F}$ is a conservative vector field, the amount of work done while moving a particle from $(1,2,1)$ to $(2,1,1)$ is given by

$$\begin{aligned} W &= \phi(2,1,1) - \phi(1,2,1) \\ &= 4\ln 1 - 2 - \ln 2 + 2 \\ &= -\ln 2. \end{aligned}$$

Problem 5: *(15 points)* Let S be the "cup" surface formed by rotating the curve $x = \sin z, y = 0, 0 \le z \le \pi/2$ around the $z-$axis. Find the mass of the surface, where the density is given by $\delta(x, y, z) = \sqrt{1 - x^2 - y^2}$.

Solution: A parametric equation of the "cup" surface is

$$\vec{R}(z,\theta) \quad = \quad \sin z \cos\theta \hat{i} + \sin z \sin\theta \hat{j} + z\hat{k}, \quad 0 \le z \le \pi/2, 0 \le \theta \le 2\pi.$$

The mass of the surface is given by

$$M = \int\int \delta \, d\sigma,$$

where $d\sigma = \left| \frac{\partial \vec{R}}{\partial z} \times \frac{\partial \vec{R}}{\partial \theta} \right| \, dz d\theta$.

$$\begin{aligned}
\frac{\partial \vec{R}}{\partial z} &= \cos z \cos\theta \hat{i} + \cos z \sin\theta \hat{j} + \hat{k} \\
\frac{\partial \vec{R}}{\partial \theta} &= -\sin z \sin\theta \hat{i} + \sin z \cos\theta \hat{j} \\
\frac{\partial \vec{R}}{\partial z} \times \frac{\partial \vec{R}}{\partial \theta} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos z \cos\theta & \cos z \sin\theta & 1 \\ -\sin z \sin\theta & \sin z \cos\theta & 0 \end{vmatrix} \\
&= (0 - \sin z \cos\theta)\hat{i} - (\sin z \sin\theta)\hat{j} + (\cos z \sin z)\hat{k} \\
\left| \frac{\partial \vec{R}}{\partial z} \times \frac{\partial \vec{R}}{\partial \theta} \right| &= \sqrt{(-\sin z \cos\theta)^2 + (\sin z \sin\theta)^2 + (\cos z \sin z)^2} \\
&= \sqrt{\sin^2 z + (\cos z \sin z)^2} \\
&= \sin z \sqrt{1 + \cos^2 z} \quad (\text{since } 0 \le z \le \pi/2, \, \sin z \ge 0).
\end{aligned}$$

The density of the surface is given by

$$\begin{aligned}
\delta &= \sqrt{1 - x^2 - y^2} \\
&= \sqrt{1 - \sin^2 z \cos^2\theta - \sin^2 z \sin^2\theta} \\
&= \sqrt{1 - \sin^2 z} \\
&= |\cos z| \\
&= \cos z \quad (\text{since } 0 \le z \le \pi/2, \, \cos z \ge 0).
\end{aligned}$$

Therefore mass of the surface is

$$\begin{aligned}
M &= \int_0^{2\pi} \int_0^{\pi/2} \cos z \sin z \sqrt{1 + \cos^2 z} \, dz d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3}(1 + \cos^2 z)^{3/2} \Big|_0^{\pi/2} \right] \, d\theta \\
&= \frac{1}{3}(2\sqrt{2} - 1) \int_0^{2\pi} \, d\theta \\
&= \frac{2\pi}{3}(2\sqrt{2} - 1).
\end{aligned}$$

Alternative Method

We know that if we rotate the graph of $x = f(z)$ with respect to the z-axis, then it creates a surface. If we cut that surface by the horizontal plane $z = c$, we obtain a circle of radius $f(c)$. Equation of that circle is $x^2 + y^2 = (f(c))^2, z = c$. Since this is true for any z, the equation of the surface of revolution is $x^2+y^2 = (f(z))^2$. In our case, the equation of the surface is $x^2+y^2 = \sin^2 z$, $0 \le z \le \pi/2$. Which can also be written as $F(x, y, z) = 0$, where $F(x, y, z) = x^2+y^2-\sin^2 z$, $0 \le z \le \pi/2$. So $\vec{\nabla} F = 2x\hat{i}+2y\hat{j}-(2\sin z\cos z)\hat{k}$. Mass of the surface is

$$M = \int\int \delta \, d\sigma$$

The projection of this surface on the xy-plane is the unit circle of radius 1. So

$$\begin{aligned}
M &= \int\int_{x^2+y^2\le 1} \sqrt{1-x^2-y^2}\frac{|\vec{\nabla}F|}{|\vec{\nabla}F\cdot\hat{k}|}\,dxdy \\
&= \int\int_{x^2+y^2\le 1} \sqrt{1-x^2-y^2}\frac{2\sqrt{x^2+y^2+\sin^2 z\cos^2 z}}{|2\sin z\cos z|}\,dxdy \\
&= \int\int_{x^2+y^2\le 1} |\cos z|\frac{|\sin z|\sqrt{1+\cos^2 z}}{|\sin z\cos z|}\,dxdy \quad (\text{since } x^2+y^2=\sin^2 z) \\
&= \int\int_{x^2+y^2\le 1} \sqrt{2-x^2-y^2}\,dxdy \quad (\text{since } x^2+y^2=\sin^2 z) \\
&= \int_0^{2\pi}\int_0^1 \sqrt{2-r^2}\,rdrd\theta \\
&= \int_0^{2\pi}\left[-\frac{1}{3}(2-r^2)^{3/2}\Big|_0^1\right]d\theta \\
&= \frac{2\pi}{3}(2\sqrt{2}-1).
\end{aligned}$$

Problem 6:*(15 points)* Let $\hat{n}$ be the outer unit normal (normal away from the origin) of the parabolic shell $S: \quad 4x^2 + y + z^2 = 4,\ y \geq 0$, and let

$$\vec{F} = \left(-z + \frac{1}{2+x}\right)\hat{\imath} + (\tan^{-1} y)\hat{\jmath} + \left(x + \frac{1}{4+z}\right)\hat{k}.$$

Find the value of

$$\int\int_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma.$$

Solution: The boundary of the parabolic shell lies on the $xz-$plane. And the equation of the boundary is $4x^2 + z^2 = 4$ i.e., $x^2 + (z/2)^2 = 1$. A parametric equation of this boundary (in the counterclockwise orientation) is

$$C : \vec{r}(t) = \sin t\hat{\imath} + 2\cos t\hat{k}, \quad 0 \leq t \leq 2\pi.$$

Using the Stokes' theorem we have

$$\begin{aligned}
\int\int_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma &= \oint_C \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt \\
&= \oint_C \left[\left(-z + \frac{1}{2+x}\right)\cos t + \left(x + \frac{1}{4+z}\right)(-2\sin t)\right] dt \\
&= \int_0^{2\pi} \left[\left(-2\cos t + \frac{1}{2+\sin t}\right)\cos t + \left(\sin t + \frac{1}{4+2\cos t}\right)(-2\sin t)\right] dt \\
&= \int_0^{2\pi} \left[-2 + \frac{\cos t}{2+\sin t} - \frac{2\sin t}{4+2\cos t}\right] dt \\
&= -2t + \ln|2+\sin t| + \ln|4+2\cos t|\Big|_0^{2\pi} \\
&= -4\pi.
\end{aligned}$$

Extra Credit: *(2 points)* I'll think about it.