MAT 21D Max time: 1 hr 20 mins **Practice Final 1 (Solutions)**

Name:____ September 10, 2015

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).
- You may bring hand written notes ONLY ON ONE SIDE of a half page (where full page = max A4).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- Check that you have all **eight** pages of the exam. (The number of pages includes this cover page.)
- There is an *extra credit problem* on the last page.

During the exam:

- Keep your eyes on your own exam!

Note that the exam length is exactly 1 hr 20 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	5	6	**Total**
Score							

Problem 1:*(10 points)* Evaluate the following integral

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx.$$

Solution: Reversing the order of integration

$$\begin{aligned}
\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dxdy \\
&= \int_0^4 \frac{e^{2y}}{4-y} \left[\frac{x^2}{2} \Big|_0^{\sqrt{4-y}} \right] dy \\
&= \frac{1}{2} \int_0^4 e^{2y}\, dy \\
&= \frac{1}{4} e^{2y} \Big|_0^4 \\
&= \frac{1}{4}(e^8 - 1).
\end{aligned}$$

Problem 2: *(10 points)* Find the mass of the solid spherical cap $x^2 + y^2 + z^2 \leq 25, z \geq 3$ *[which is obtained by cutting the solid sphere* $x^2 + y^2 + z^2 \leq 25$ *by the plane* $z = 3$*]*. The density of the cap is given by $\delta = 1$.

Solution: The plane $z = 3$ cuts the sphere $x^2 + y^2 + z^2 \leq 25$ along the circle $x^2 + y^2 + 3^2 \leq 25$ i.e., $x^2 + y^2 \leq 16$. In the cylindrical coordinate system, the mass of spherical cap is given by the following integral

$$
\begin{aligned}
M &= \int_0^{2\pi} \int_0^4 \int_3^{\sqrt{25-r^2}} 1 \cdot dz\, r dr d\theta \\
&= \int_0^{2\pi} \int_0^4 (\sqrt{25-r^2} - 3) r dr d\theta \\
&= \int_0^{2\pi} -\frac{1}{3}(25-r^2)^{3/2} - \frac{3}{2}r^2 \Big|_0^4 \, d\theta \\
&= \int_0^{2\pi} \left(-\frac{1}{3}9^{3/2} - 24 + \frac{1}{3}(25)^{3/2}\right) d\theta \\
&= \int_0^{2\pi} \left(-\frac{27}{3} - 24 + \frac{125}{3}\right) d\theta \\
&= \int_0^{2\pi} \frac{26}{3}\, d\theta \\
&= \frac{52\pi}{3}.
\end{aligned}
$$

Problem 3: *(15 points)* Consider the following space curve

$$\vec{r}(t) = (e^t \cos t)\ \hat{\imath} + (e^t \sin t)\ \hat{\jmath} + \sqrt{2}e^t\ \hat{k}.$$

(a) Find the curvature κ of above space curve at the point $t = 0$.

(b) Find the tangential component a_T and the normal component a_N of acceleration at $t = 0$.

Solution: (a) Differentiating the parametric equation of the curve with respect to t we obtain

$$\begin{aligned}
\vec{v}(t) &= e^t(\cos t - \sin t)\ \hat{\imath} + e^t(\sin t + \cos t)\ \hat{\jmath} + \sqrt{2}e^t\ \hat{k} \\
|\vec{v}(t)| &= e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 2} = 2e^t \\
\vec{T}(t) &= \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{2}(\cos t - \sin t)\ \hat{\imath} + \frac{1}{2}(\sin t + \cos t)\ \hat{\jmath} + \frac{\sqrt{2}}{2}\ \hat{k} \\
\frac{d\vec{T}}{dt} &= \frac{1}{2}(-\sin t - \cos t)\ \hat{\imath} + \frac{1}{2}(\cos t - \sin t)\ \hat{\jmath} \\
\left|\frac{d\vec{T}}{dt}\right| &= \frac{1}{2}\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2} = \frac{\sqrt{2}}{2} \\
\kappa(t) &= \frac{|d\vec{T}/dt|}{|\vec{v}(t)|} = \frac{\sqrt{2}}{4e^t}.
\end{aligned}$$

Therefore the curvature at $t = 0$ is $\kappa(0) = \sqrt{2}/4$.

(b) Tangential and the normal components of acceleration are given by

$$\begin{aligned}
a_T(t) &= \frac{d}{dt}|\vec{v}(t)| = 2e^t, \\
a_N(t) &= \kappa(t)|\vec{v}(t)|^2 = \sqrt{2}e^t.
\end{aligned}$$

Therefore the tangential and the normal components of acceleration at $t = 0$ are given by $a_T(0) = 2,\ a_N(0) = \sqrt{2}$.

Problem 4: *(15 points)*

Consider the vector fields $\vec{F}_1$, $\vec{F}_2$, and the curve C given below

$$\vec{F}_1 = (3x^2y^2 + 2y)\,\hat{i} + (2x^3y - 3x)\,\hat{j},$$
$$\vec{F}_2 = (2y\cos x + 3x)\,\hat{i} + (y^2\sin x + 2y)\,\hat{j},$$
$$C:\ \vec{r}(t) = \cos^3 t\,\hat{i} + \sin^3 t\,\hat{j}, \quad 0 \le t \le 2\pi$$

(a) Find the area of the region enclosed by the curve C.

(b) Find the counter-clockwise circulation of the vector field $\vec{F}_1$ along the curve C.

(c) Find the outward flux of the vector field $\vec{F}_2$ across the curve C.

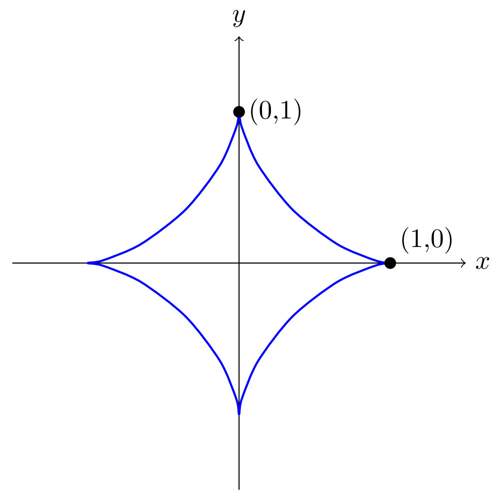


Figure 1: Graph of the curve C.

Solution: (a) From Green's theorem, we know that the area enclosed by a curve $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{t}$ is given by $\frac{1}{2}\oint(x\,dy - y\,dx)$. Therefore, in our case

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_0^{2\pi}\left[(\cos^3 t)(3\sin^2 t\cos t) - (\sin^3 t)(-3\cos^2 t\sin t)\right]\,dt \\
&= \frac{3}{2}\int_0^{2\pi}\left[\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right]\,dt \\
&= \frac{3}{2}\int_0^{2\pi}\cos^2 t\sin^2 t\,dt \\
&= \frac{3}{8}\int_0^{2\pi}\sin^2 2t\,dt \\
&= \frac{3}{16}\int_0^{2\pi}(1-\cos 4t)\,dt \\
&= \frac{3\pi}{8}.
\end{aligned}$$

(b) Let S be the region enclosed by the curve C. Using the Green's theorem, we can compute the circulation as

$$\begin{aligned}
\int\int_S\left[\frac{\partial}{\partial x}(2x^3y - 3x) - \frac{\partial}{\partial y}(3x^2y^2 + 2y)\right]\,dA &= \int\int_S[(6x^2y - 3) - (6x^2y + 2)]\,dA \\
&= \int\int_S(-5)\,dA \\
&= (-5)\times\text{Area of } S = -\frac{15\pi}{8}.
\end{aligned}$$

(c) Using the Green's theorem, we can compute the flux as

$$\begin{aligned}
\int\int_S\left[\frac{\partial}{\partial x}(2y\cos x + 3x) + \frac{\partial}{\partial y}(y^2\sin x + 2y)\right]\,dA &= \int\int_S[(-2y\sin x + 3) + (2y\sin x + 2)]\,dA \\
&= \int\int_S 5\,dA = \frac{15\pi}{8}.
\end{aligned}$$

Problem 5: *(15 points)* Find the moment of inertia of the thin spherical shell $x^2+y^2+z^2=a^2$ with respect to any diameter of it. The density of the shell is $\delta = 1$.

Solution: Since it is a symmetric object, the moment of inertia is same for any diameter. Let us find the moment of inertia with respect to the $z-$axis.

$$I_z = \int\int_S (x^2+y^2)\,\delta\, d\sigma.$$

To find $d\sigma$, we notice that the equation of the spherical shell is $f(x,y,z)=0$, where $f(x,y,z)=x^2+y^2+z^2-a^2$. Taking the projection onto the xy-plane,

$$\begin{aligned} d\sigma &= 2\frac{|\vec{\nabla} f|}{|\vec{\nabla} f\cdot \hat{k}|}\, dxdy \\ &= 2\frac{|2x\hat{i}+2y\hat{j}+2z\hat{k}|}{|2z|}\, dxdy \\ &= 2\frac{\sqrt{x^2+y^2+z^2}}{|z|}\, dxdy \\ &= \frac{2a}{\sqrt{a^2-x^2-y^2}}\, dxdy. \end{aligned}$$

We multiplied the above by 2, because the projection of the lower hemisphere overlaps with the projection of the upper hemisphere. Since the projection of the hemisphere on the $xy-$plane is the circle $x^2+y^2\le a^2$,

$$\begin{aligned} I_z &= \int\int_{x^2+y^2\le a^2} (x^2+y^2)\frac{2a}{\sqrt{a^2-x^2-y^2}}\, dxdy \\ &= \int_0^{2\pi}\int_0^a r^2\frac{2a}{\sqrt{a^2-r^2}}\, rdrd\theta \quad \text{(changing it to the polar coordinates)} \\ &= a\int_0^{2\pi}\int_0^{a^2} (a^2-u)\frac{1}{\sqrt{u}}\, dud\theta \quad \text{(substituting } a^2-r^2=u\text{)} \\ &= a\int_0^{2\pi}\int_0^{a^2} (a^2u^{-1/2}-u^{1/2})\, dud\theta \\ &= a\int_0^{2\pi} 2a^2u^{1/2}-\frac{2}{3}u^{3/2}\Big|_0^{a^2}\, d\theta \\ &= a\int_0^{2\pi} \frac{4a^3}{3}\, d\theta \\ &= \frac{8\pi a^4}{3}. \end{aligned}$$

Problem 6:*(15 points)* Consider the ellipsoidal shell $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and let

$$\vec{F} = \frac{y^2 x}{b^2}\hat{i} + \frac{z^2 y}{c^2}\hat{j} + \frac{x^2 z}{a^2}\hat{k},$$

where a, b, c are constants. Find the outward flux of $\vec{F}$ across the surface of the ellipsoidal shell.

Solution: Since we have a closed surface, we can use the divergence theorem. Let us denote the ellipsoidal surface by S and the solid ellipsoid is E. The flux of the vector field across the ellipsoidal surface is

$$\int\int_S \vec{F}\cdot\hat{n}\, d\sigma \quad = \quad \int\int\int_E \vec{\nabla}\cdot\vec{F}\, dv.$$

Divergence of the vector field $\vec{F}$ is

$$\begin{aligned} \vec{\nabla}\cdot\vec{F} &= \frac{\partial}{\partial x}\left(\frac{y^2 x}{b^2}\right) + \frac{\partial}{\partial y}\left(\frac{z^2 y}{c^2}\right) + \frac{\partial}{\partial z}\left(\frac{x^2 z}{z^2}\right) \\ &= \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{x^2}{a^2}. \end{aligned}$$

So the flux is

$$\int\int\int_E \vec{\nabla}\cdot\vec{F}\, dv \quad = \quad \int\int\int_E \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\, dxdydz.$$

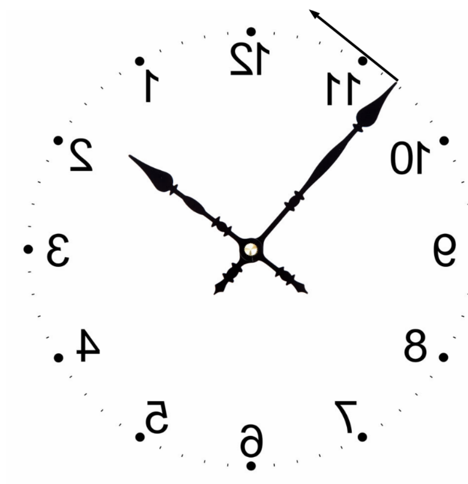
Let us substitute $x = au, y = bv, z = cw$. Then the ellipsoid becomes a unit sphere $US: \quad u^2 + v^2 + w^2 = 1$. And the jacobian of the transformation is

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \\ &= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \\ &= abc \end{aligned}$$

Therefore

$$\begin{aligned} \int\int\int_E \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\, dxdydz &= abc\int\int\int_{US}(u^2 + v^2 + w^2)\, dudvdw \\ &= abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\cdot\rho^2\sin\phi\, d\rho d\phi d\theta \quad \text{(changing it to} \\ &\quad \text{the spherical coordinate system)} \\ &= \frac{abc}{5}\int_0^{2\pi}\int_0^{\pi}\sin\phi d\phi d\theta \\ &= \frac{abc}{5}\int_0^{2\pi} -\cos\phi|_0^{\pi}\ d\theta \\ &= \frac{abc}{5}\int_0^{2\pi} 2\ d\theta \\ &= \frac{4\pi abc}{5}. \end{aligned}$$

Extra Credit:*(2 points)*



The arrow shows the direction of motion of the minute hand. Is the minute hand rotating clockwise or counter-clockwise?