MAT 21D Max time: 1 hr 20 mins **Midterm**

Name:____ August 25, 2016

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).
- You may bring hand written notes ONLY ON ONE SIDE of a half page (where full page = max A4).

As soon as the exam starts:

- Take a quick breath to relax! If you have worked through all the homework problems then you will do fine!
- Check that you have all **eight** pages of the exam. (The number of pages includes this cover page.)
- There is an *extra credit problem* on the last page.

During the exam:

- Keep your eyes on your own exam!

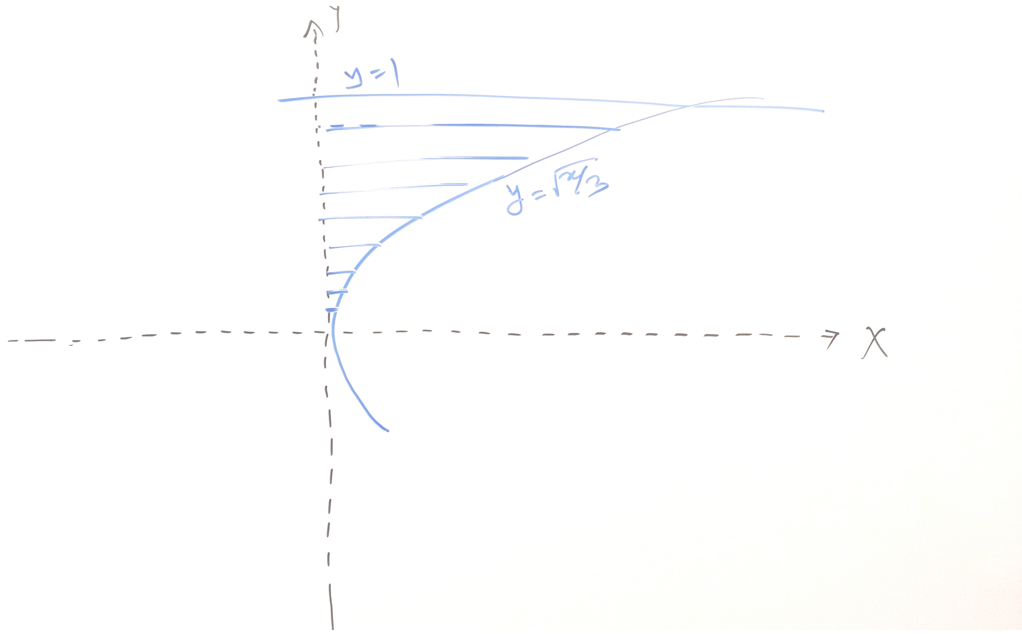
Note that the exam length is exactly 1 hr 20 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	5	6	**Total**
Score							

Problem 1:*(10 points)* Sketch the region of integration, and then evaluate the following integral

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\ dydx.$$

Solution: Solution:



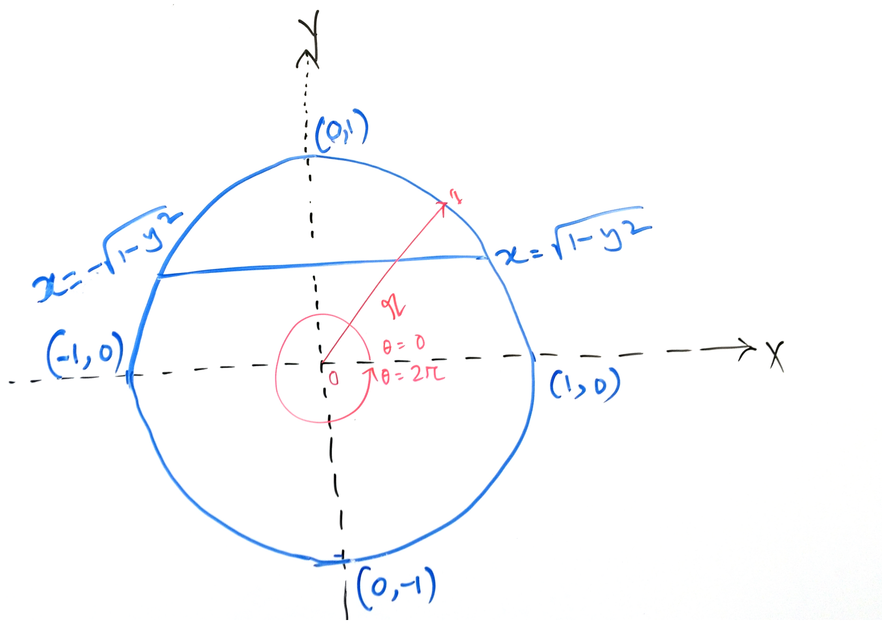
Using the picture and reversing the order of integration we have

$$\begin{aligned}\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\ dydx &= \int_0^1 \int_0^{3y^2} e^{y^3}\ dxdy \\ &= \int_0^1 3y^2 e^{y^3}\ dy \\ &= e^{y^3}\Big|_0^1 \\ &= e-1.\end{aligned}$$

Problem 2: *(10 points)* Evaluate the following integral

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{(1+x^2+y^2)^2}\, dxdy.$$

Solution:



Converting the integration into the polar coordinate system, we have

$$\begin{aligned}
\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{(1+x^2+y^2)^2}\, dxdy &= \int_0^{2\pi} \int_0^1 \frac{2}{(1+r^2)^2} r\, drd\theta \\
&= \int_0^1 \left[-\frac{1}{(1+r^2)} \bigg|_0^1 \right] d\theta \\
&= \left[-\frac{1}{2} + 1 \right] 2\pi \\
&= \pi.
\end{aligned}$$

Problem 3: *(15 points)* Consider the helix $\vec{r}(t) = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + bt\,\hat{k}$, $a, b > 0$.

(a) Find the unit tangent vector $\vec{T}$, unit normal vector $\vec{N}$, and the curvature κ of this helix.

(b) What happens to κ when we take $a \to \infty$ or $b \to \infty$? What is the geometric reason behind this behaviour?

Solution:

(a) The equation of the space curve is given by

$$\vec{r}(t) = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + bt\,\hat{k}.$$

From this equation we can compute the following quantities.

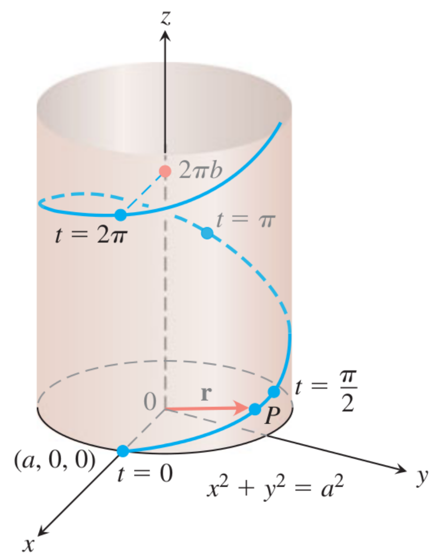


Figure 1: The blue line indicates the graph of the helix.

$$\begin{aligned}
\vec{v}(t) &= \frac{d\vec{r}}{dt} = (-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + b\,\hat{k} \\
|\vec{v}(t)| &= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2+b^2} \\
\vec{T} &= \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{\sqrt{a^2+b^2}}\Big[(-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + b\,\hat{k}\Big] \\
\frac{d\vec{T}}{dt} &= \frac{1}{\sqrt{a^2+b^2}}\Big[(-a\cos t)\,\hat{i} + (-a\sin t)\,\hat{j}\Big] \\
\left|\frac{d\vec{T}}{dt}\right| &= \frac{a}{\sqrt{a^2+b^2}} \\
\vec{N} &= \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = -\cos t\,\hat{i} - \sin t\,\hat{j} \\
\kappa &= \frac{|d\vec{T}/dt|}{|\vec{v}(t)|} = \frac{a}{a^2+b^2}.
\end{aligned}$$

(b) From the expression of κ, we notice that

$$\lim_{a\to\infty} \kappa = 0 = \lim_{b\to\infty} \kappa.$$

When we increase a, the radius of the helix gets bigger. Therefore as $a \to \infty$, locally it looks like a line. Since the curvature of a line is zero, κ becomes zero.

When we increase b, the helix is being stretched out vertically. Therefore as $b \to \infty$, it becomes a line, and consequently the curvature κ becomes zero.

Problem 4: *(15 points)*

(a) Find the mass of the thick spherical shell (S_1) which is trapped inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the sphere $x^2 + y^2 + z^2 = 4$. Density of the shell is given by $\delta(x, y, z) = x^2 + y^2 + z^2$.

(b) Find the mass of another solid sphere (S_2) $x^2 + y^2 + z^2 = 9$, whose density is given by $\delta(x, y, z) = \frac{1}{45}(3^5 - 2^5) = \frac{211}{45}$.

(c) Which one of S_1 and S_2 has the higher moment of inertia with respect to their diameters? Explain your answer.

Solution:

(a) Mass of the thick spherical shell S_1 is

$$
\begin{aligned}
M_1 &= \int\int\int_{S_1} \delta(x, y, z)\, dV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_2^{3} \rho^2 \times \rho^2 \sin\phi \; d\rho d\phi d\theta \\
&= \frac{1}{5}(3^5 - 2^5)\int_0^{2\pi}\int_0^{\pi} \sin\phi \; d\phi d\theta \\
&= \frac{1}{5}(3^5 - 2^5)\int_0^{2\pi} [-\cos\phi|_0^{\pi}] \; d\theta \\
&= \frac{1}{5}(3^5 - 2^5) \times 2 \times 2\pi = \frac{4\pi}{5}(3^5 - 2^5).
\end{aligned}
$$

(b) Since the density of this sphere S_2 is constant, mass of this sphere can easily be computed by

$$
\begin{aligned}
M_2 &= \text{volume} \times \text{density} \\
&= \frac{4\pi}{3} \times 3^3 \times \frac{1}{45}(3^5 - 2^5) \\
&= \frac{4\pi}{5}(3^5 - 2^5).
\end{aligned}
$$

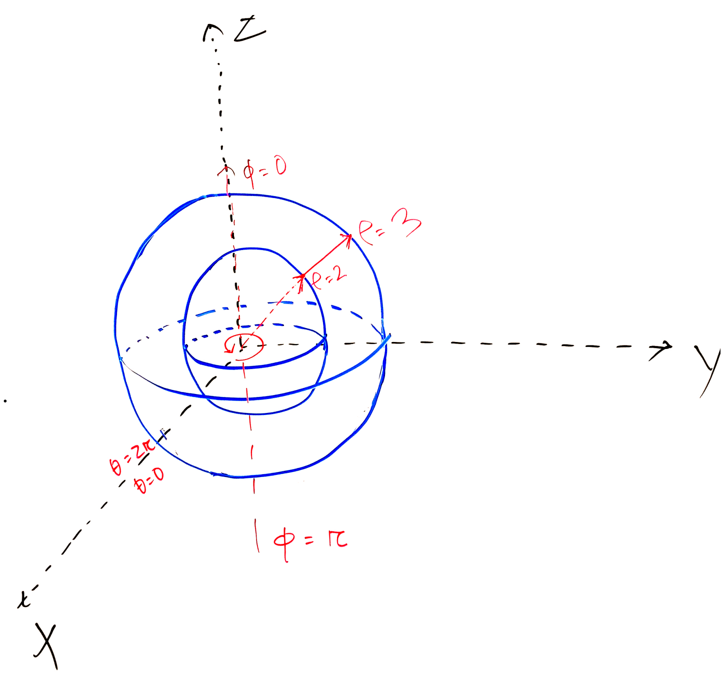


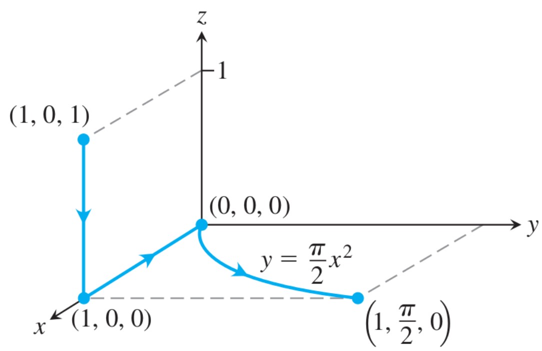
Figure 2: Picture of the thick Shell S_1.

(c) We notice that mass and the radius of both of the spheres are the same. However, since the density is constant for S_2, the mass is uniformly distributed over S_2. Whereas the density of the first sphere S_1 is increasing towards the outer shell, and there is no mass in the hollow part $x^2 + y^2 + z^2 \leq 4$. So, overall the mass of the first sphere is distributed away from the origin. Consequently, S_1 will have higher moment of inertia with respect to it's diameter than S_2.

Problem 5:*(15 points)* In a stormy weather, a fly is trying to reach it's home which is located at $(1, \pi/2, 0)$ from it's current location $(1, 0, 1)$. If the effective wind force on the fly is given by

$$\vec{F}(x,y,z) = e^{yz}\hat{\imath} + (xze^{yz} + z\cos y)\hat{\jmath} + (xye^{yz} + \sin y)\hat{k},$$

and the fly takes the path shown below, find the work done by the fly to reach it's home.



[Hint: Is $\vec{F}$ a conservative force field?]

Solution: Let us check whether the vector field is conservative or not.

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{yz} & xze^{yz} + z\cos y & xye^{yz} + \sin y \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(xye^{yz} + \sin y) - \frac{\partial}{\partial z}(xze^{yz} + z\cos y)\right]\hat{\imath} + \left[\frac{\partial}{\partial z}(e^{yz}) - \frac{\partial}{\partial x}(xye^{yz} + \sin y)\right]\hat{\jmath} \\
&\quad + \left[\frac{\partial}{\partial x}(xze^{yz} + z\cos y) - \frac{\partial}{\partial y}(e^{yz})\right]\hat{k} \\
&= \left[(xe^{yz} + xyze^{yz} + \cos y) - (xe^{yz} + xyze^{yz} + \cos y)\right]\hat{\imath} + \left[ye^{yz} - ye^{yz}\right]\hat{\jmath} \\
&\quad + \left[ze^{yz} - ze^{yz}\right]\hat{k} \\
&= \vec{0}.
\end{aligned}$$

Therefore the vector field $\vec{F}$ is conservative. Consequently, the work done by the fly does not depend on the path of travel. It depends only on the initial and the final point. Since $\vec{F}$ is conservative, there exists a potential function f such that $\vec{F} = \vec{\nabla} f$ and the work done by the fly is

$$f(1, \pi/2, 0) - f(1, 0, 1).$$

To find the the potential function f,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= e^{yz} & &\Rightarrow f(x,y,z) = xe^{yz} + C_1(y,z) \\
\frac{\partial f}{\partial y} &= xze^{yz} + z\cos y & &\Rightarrow f(x,y,z) = xe^{yz} + z\sin y + C_2(x,z) \\
\frac{\partial f}{\partial z} &= xye^{yz} + \sin y & &\Rightarrow f(x,y,x) = xe^{yz} + z\sin y + C_3(x,y).
\end{aligned}$$

From the above three equations, we conclude that $f(x, y, z) = xe^{yz} + z \sin y + C$. Therefore the work done by the fly is

$$f(1, \pi/2, 0) - f(1, 0, 1) = (e^0 + 0 \sin \pi/2) - (e^0 + \sin 0) = 0.$$

Problem 6: *(15 points)* Consider the 'truncated horizontal ice-cream cone' which is trapped inside the sphere $x^2+y^2+z^2=4$, outside the sphere $x^2+y^2+z^2=1$ and inside the cone $x=\sqrt{y^2+z^2}$. Density of the solid is given by $\delta(x,y,z)=1/(x^2+y^2+z^2)$. Find the centroid of this solid.

Solution:

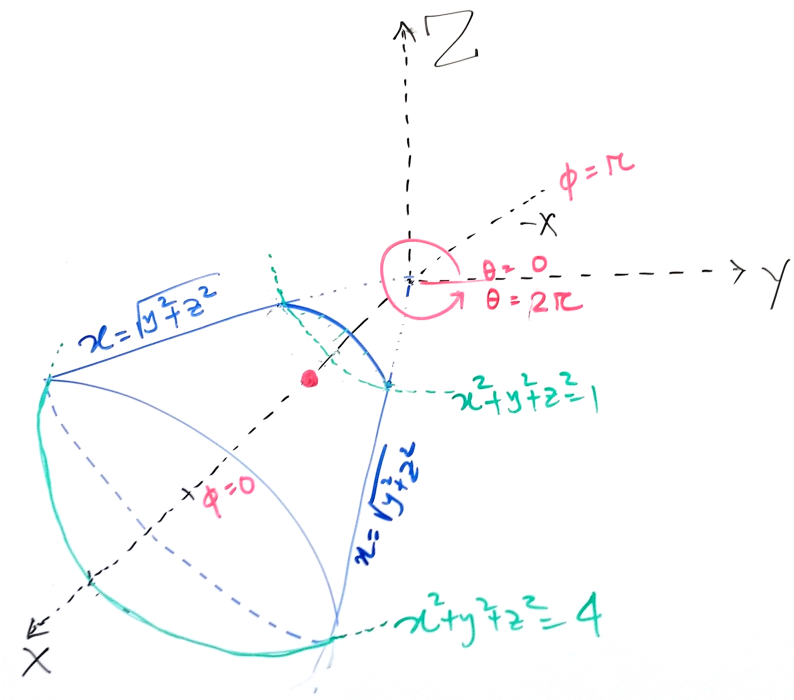


Figure 3: Picture of the solid. The red dot indicates the possible position of the centroid.

Since the solid and the density are symmetric with respect to the x-axis. Therefore the centroid must lie on the x-axis. In other words the centroid has the form $(\bar{x},0,0)$.

Mass of the solid is given by $M=\int\int\int \delta(x,y,z)\ dV$. Since the solid is symmetric with respect to x-axis, and it is spherically symmetric, it is convenient to use the spherical coordinate system by taking x-axis as the $\phi=0$ line, yz-plane as the plane of θ, and y-axis as the $\theta=0$ line. In this coordinate system, we have

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi\cos\theta \\ z &= \rho\sin\phi\sin\theta. \end{aligned}$$

Then mass of the solid is given by

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \frac{1}{\rho^2}\rho^2\sin\phi\ d\rho d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/4}\sin\phi\ d\phi d\theta \\ &= \int_0^{2\pi}\left[-\cos\phi|_0^{\pi/4}\right]\ d\theta = 2\pi\left[1-\frac{\sqrt{2}}{2}\right]. \end{aligned}$$

The moment of the solid with respect to the yz-plane is given by

$$\begin{aligned}
M_{yz} &= \int\int\int x\delta(x,y,z)\,dV \\
&= \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho\cos\phi\frac{1}{\rho^2}\rho^2\sin\phi\ d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{1}{2}\rho^2\Big|_1^2\right]\cos\phi\sin\phi\ d\phi d\theta \\
&= \frac{3}{2}\int_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\Big|_0^{\pi/4}\right]\ d\theta \\
&= \frac{3}{2}\times\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^2\times 2\pi = \frac{3\pi}{4}.
\end{aligned}$$

Therefore the x coordinate of the centroid of this solid is

$$\bar{x} = \frac{M_{yz}}{M} = \frac{3\pi/4}{2\pi[1-\sqrt{2}/2]} = \frac{3}{8\left[1-\frac{\sqrt{2}}{2}\right]}.$$

Consequently, the centroid is

$$\left(\frac{3}{8\left[1-\frac{\sqrt{2}}{2}\right]}, 0, 0\right).$$