MAT 21D Max time: 1 hr 20 mins **Final**

Name:______________________________ September 8, 2016

Before the exam begins:

- Write your name above.
- Turn off all electronics and keep them out of sight: no calculator, no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).
- You may bring hand written notes ONLY ON ONE SIDE of a half page (where full page = max A4).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- Check that you have all **eight** pages of the exam. (The number of pages includes this cover page.)
- There is an *extra credit problem* on the last page.

During the exam:

- Keep your eyes on your own exam!

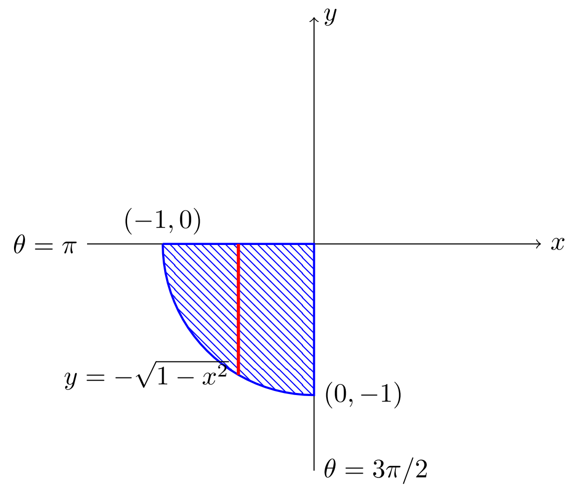
Note that the exam length is exactly 1 hr 20 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	5	6	**Total**
Score							

Problem 1:*(10 points)* Change the following Cartesian integral to an equivalent polar form. Then evaluate the integral

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{1}{1+\sqrt{x^2+y^2}}\,dy\,dx.$$

Solution:



From the picture, we see that we can convert the integral to polar form as follows

$$\begin{aligned}
\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{1}{1+\sqrt{x^2+y^2}}\,dy\,dx &= \int_{\pi}^{3\pi/2}\int_{0}^{1} \frac{r\,dr\,d\theta}{1+r} \\
&= \int_{\pi}^{3\pi/2}\int_{0}^{1} \frac{(1+r)-1}{1+r}\,dr\,d\theta \\
&= \int_{\pi}^{3\pi/2}\int_{0}^{1} \left[1-\frac{1}{1+r}\right]\,dr\,d\theta \\
&= \int_{\pi}^{3\pi/2} \left[r-\ln(1+r)\Big|_0^1\right]\,d\theta \\
&= (1-\ln 2)\pi/2.
\end{aligned}$$

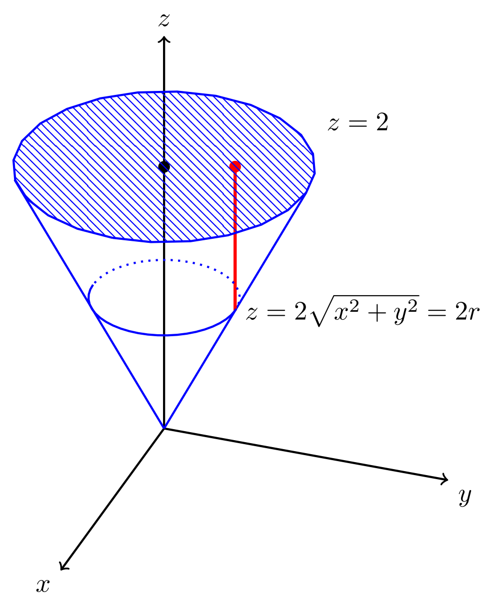
Problem 2: *(10 points)* Consider the *solid* cone $z = 2\sqrt{x^2+y^2}$ which is bounded above by the plane $z = 2$. Density of the cone is given by $\delta(x, y, z) = z$. Find the centroid of this cone.

Solution:

Note that the top of the cone is located on the $z = 2$ plane, and it is the circle $2 = 2\sqrt{x^2+y^2}$ i.e., $x^2 + y^2 = 1$.

We observe that the cone and the density of the cone is symmetric with respect to z axis. Therefore the centroid must lie on z-axis. In other words, the centroid is of the form $(0, 0, \bar{z})$.

Using the cylindrical coordinate system, we can find the mass of the cone as



$$\begin{aligned}
M &= \int_0^{2\pi}\int_0^1\int_{2r}^2 z\,dz\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left[\frac{1}{2}z^2\Big|_{2r}^{2}\right] r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 (2-2r^2)r\,dr\,d\theta \\
&= \int_0^{2\pi} \left[r^2 - \frac{2}{4}r^4\Big|_0^1\right] d\theta \\
&= 2\pi \times (1 - 1/2) = \pi.
\end{aligned}$$

Moment of the cone with respect to the xy-plane is

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi}\int_0^1\int_{2r}^2 z \times z\,dz\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left[\frac{z^3}{3}\Big|_{2r}^{2}\right] r\,dr\,d\theta \\
&= \frac{8}{3}\int_0^{2\pi}\int_0^1 (1-r^3)r\,dr\,d\theta \\
&= \frac{8}{3}\int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^5}{5}\Big|_0^1\right] d\theta \\
&= \frac{8}{3} \times (1/2 - 1/5) \times 2\pi \\
&= \frac{8}{3} \times \frac{3}{10} \times 2\pi = \frac{8\pi}{5}.
\end{aligned}$$

Therefore

$$\bar{z} = \frac{M_{xy}}{M} = \frac{8\pi/5}{\pi} = \frac{8}{5}.$$

So the centroid of the cone is $(0, 0, 8/5)$

Problem 3: *(15 points)* Suppose a particle is moving along the following parabola

$$\vec{r}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath}, \quad -\infty < t < \infty.$$

(a) Find the normal component of acceleration a_N of the particle.

* *[Notice that a_N is a function of t.]*

(b) Show that a_N is maximized at the vertex of the parabola.

Solution: (a) Differentiating the equation of the curve with respect to t, we have

$$\begin{aligned} \vec{v}(t) &= \hat{\imath} + 2t\,\hat{\jmath} \\ \vec{a}(t) &= \frac{d\vec{v}}{dt} = 2\,\hat{\jmath} \\ |\vec{v}(t)| &= \sqrt{1+4t^2}. \end{aligned}$$

Therefore the tangential component is given by

$$\begin{aligned} a_T(t) &= \frac{d}{dt}|\vec{v}(t)| \\ &= \frac{4t}{\sqrt{1+4t^2}}. \end{aligned}$$

Consequently, the normal component is given by

$$\begin{aligned} a_N(t) &= \sqrt{|\vec{a}|^2 - a_T(t)^2} \\ &= \sqrt{4 - \frac{16t^2}{1+4t^2}} \\ &= \frac{2}{\sqrt{1+4t^2}}. \end{aligned}$$

(b) We know that $1+4t^2$ is an increasing function of $|t|$, and it is minimum at $t=0$. From the expression of $a_N(t)$, we notice that $a_N(t)$ is a decreasing function of $|t|$, and it takes the maximum value at $t=0$. Maximum value of $a_N(t)$ is $a_N(0) = 1$.

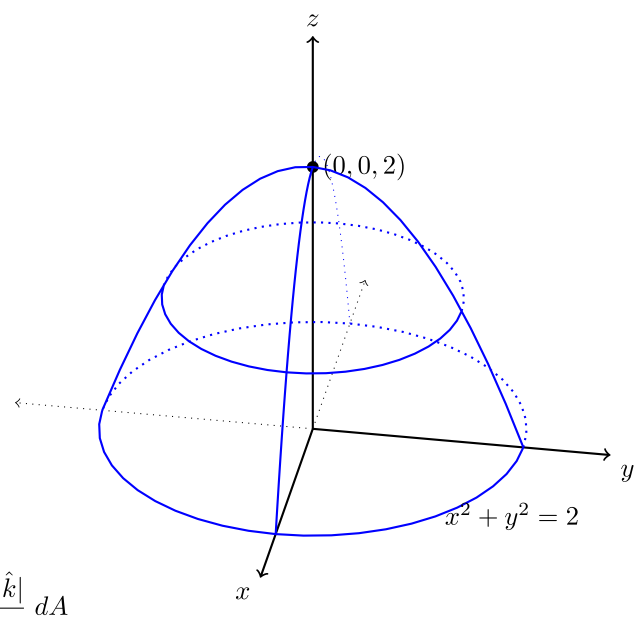
But $t=0$ corresponds to the point $(0,0)$ on $\vec{r}(t)$, which is the vertex of the parabola.

Problem 4: *(15 points)* Consider the *surface* of the parabolic shell $z = 2 - x^2 - y^2,\ z \geq 0$. Let the density of this surface is $\delta = 1$. Find the moment of inertia of this parabolic shell with respect to the z-axis.

Solution:

Let us call the surface S. Moment of inertia of the surface with respect to the z axis is

$$I_z = \iint_S (x^2 + y^2)\ \delta\ d\sigma.$$



In our case $\delta = 1$. Equation of the surface is $f(x, y, z) = x^2 + y^2 + z - 2 = 0$. From the graph, we see that the parabolic shell S is located on top of the xy-plane. It is convenient to take the shadow of S on the xy planes *(shadows taken on other planes will have overlapping)*. Notice that the shadow of S on the xy-plane is the disk $x^2 + y^2 \leq 2$. Let us call the shadow R. Then

$$\begin{aligned}
I_z &= \iint_S (x^2 + y^2)\ \delta\ d\sigma \\
&= \iint_R (x^2 + y^2) \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \hat{k}|}\ dA \\
&= \iint_R (x^2 + y^2) \frac{|2x\ \hat{\imath} + 2y\ \hat{\jmath} + \hat{k}|}{|\hat{k} \cdot \hat{k}|}\ dA \\
&= \iint_R (x^2 + y^2) \sqrt{4(x^2 + y^2) + 1}\ dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} r^2 \sqrt{1 + 4r^2}\ r\ dr\ d\theta \quad \text{(changing it to polar coordinates)} \\
&= \int_0^{2\pi} \int_1^9 \frac{u-1}{4} \sqrt{u}\ \frac{1}{8}\ du\ d\theta \quad \text{(substitution: } 1 + 4r^2 = u\text{)} \\
&= \frac{1}{32} \int_0^{2\pi} \int_1^9 (u^{3/2} - \sqrt{u})\ du\ d\theta \\
&= \frac{1}{32} \int_0^{2\pi} \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \Big|_1^9 \right]\ d\theta \\
&= \frac{\pi}{16} \left[\frac{2}{5} \times 3^5 - \frac{2}{3} \times 3^3 - \frac{2}{5} + \frac{2}{3} \right].
\end{aligned}$$

Problem 5: *(15 points)*

Consider the vector fields $\vec{F}_1$, $\vec{F}_2$, and the curve C given below

$$\vec{F}_1 = (3x^2y^2 + 2y)\ \hat{i} + (2x^3y - 3x)\ \hat{j},$$
$$\vec{F}_2 = (2y\cos x + 3x)\ \hat{i} + (y^2\sin x + 2y)\ \hat{j},$$
$$C:\ \vec{r}(t) = \cos^3 t\ \hat{i} + \sin^3 t\ \hat{j}, \quad 0 \le t \le 2\pi$$

(a) Find the area of the region enclosed by the curve C.

(b) Find the counter-clockwise circulation of the vector field $\vec{F}_1$ along the curve C.

(c) Find the outward flux of the vector field $\vec{F}_2$ across the curve C.

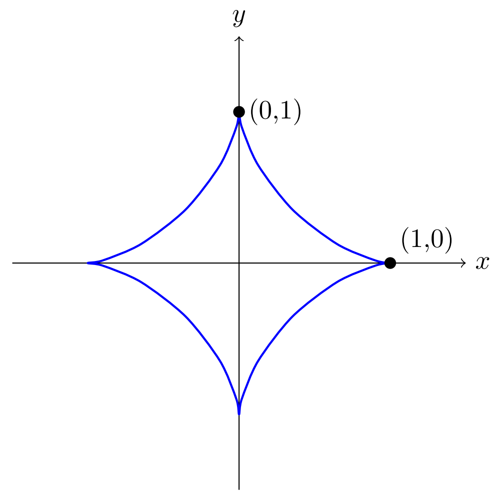


Figure 1: Graph of the curve C.

Solution: (a) From Green's theorem, we know that the area enclosed by a curve $\vec{r}(t) = x(t)\ \hat{i} + y(t)\ \hat{t}$ is given by $\frac{1}{2}\oint(x\ dy - y\ dx)$. Therefore, in our case

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_0^{2\pi} \left[(\cos^3 t)(3\sin^2 t\cos t) - (\sin^3 t)(-3\cos^2 t\sin t)\right]\ dt \\
&= \frac{3}{2}\int_0^{2\pi} \left[\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right]\ dt \\
&= \frac{3}{2}\int_0^{2\pi} \cos^2 t\sin^2 t\ dt \\
&= \frac{3}{8}\int_0^{2\pi} \sin^2 2t\ dt \\
&= \frac{3}{16}\int_0^{2\pi} (1 - \cos 4t)\ dt \\
&= \frac{3\pi}{8}.
\end{aligned}$$

(b) Let S be the region enclosed by the curve C.
Using the Green's theorem, we can compute the circulation as

$$\begin{aligned}
\int\!\!\int_S \left[\frac{\partial}{\partial x}(2x^3y - 3x) - \frac{\partial}{\partial y}(3x^2y^2 + 2y)\right]\ dA &= \int\!\!\int_S [(6x^2y - 3) - (6x^2y + 2)]\ dA \\
&= \int\!\!\int_S (-5)\ dA \\
&= (-5) \times \text{Area of } S = -\frac{15\pi}{8}.
\end{aligned}$$

(c) Using the Green's theorem, we can compute the flux as

$$\begin{aligned}
\int\!\!\int_S \left[\frac{\partial}{\partial x}(2y\cos x + 3x) + \frac{\partial}{\partial y}(y^2\sin x + 2y)\right]\ dA &= \int\!\!\int_S [(-2y\sin x + 3) + (2y\sin x + 2)]\ dA \\
&= \int\!\!\int_S 5\ dA = \frac{15\pi}{8}.
\end{aligned}$$

Problem 6: *(15 points)* Find the outward flux of the vector field $\vec{F} = (5x^3 + 12xy^2)\ \hat{\imath} + (y^3 + e^y \sin z)\ \hat{\jmath} + (5z^3 + e^y \cos z)\ \hat{k}$ across the surface of the thick sphere $1 \le x^2 + y^2 + z^2 \le 4$.

Solution: Let us call the thick sphere as D. The surface of D consists of two spherical shells, namely $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Let us call the surface as S. We know that the outward flux of $\vec{F}$ through S is given by $\oiint_S (\vec{F} \cdot \hat{n})\, d\sigma$, where $\hat{n}$ is the outward unit normal vector to the surface S.

But using the Divergence theorem, we have

$$\oiint_S (\vec{F} \cdot \hat{n})\, d\sigma = \iiint_D (\vec{\nabla} \cdot \vec{F})\, dV.$$

Divergence of the vector field $\vec{F}$ is given by

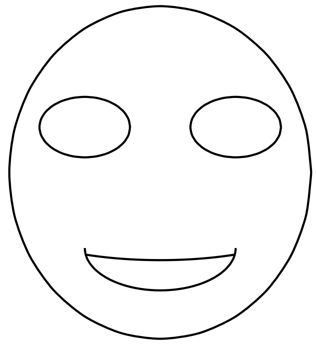
$$\begin{aligned}
\vec{\nabla} \cdot \vec{F} &= \frac{\partial}{\partial x}(5x^3 + 12xy^2) + \frac{\partial}{\partial y}(y^3 + e^y \sin z) + \frac{\partial}{\partial z}(5z^3 + e^y \cos z) \\
&= (15x^2 + 12y^2) + (3y^2 + e^y \sin z) + (15z^2 - e^y \sin z) \\
&= 15(x^2 + y^2 + z^2).
\end{aligned}$$

Therefore

$$\begin{aligned}
\iiint_D (\vec{\nabla} \cdot \vec{F})\, dV &= \iiint_D 15(x^2 + y^2 + z^2)\, dV \\
&= 15 \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^2 \times \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \quad \text{(Changing to spherical coordinates)} \\
&= 15 \int_0^{2\pi} \int_0^{\pi} \left[\frac{1}{5}\rho^5 \Big|_1^2 \right] \sin\phi\, d\phi\, d\theta \\
&= (3 \times [2^5 - 1]) \int_0^{2\pi} \left[-\cos\phi|_0^{\pi} \right]\, d\theta \\
&= 93 \times 2 \int_0^{2\pi} d\theta \\
&= 93 \times 4\pi.
\end{aligned}$$

As a result, outward flux of $\vec{F}$ across S is $93 \times 4\pi$.

Extra Credit: *(2 points)* Let $\vec{F} = M\ \hat{\imath} + N\ \hat{\jmath}$ be a conservative vector field. What is the total circulation of $\vec{F}$ along the following curves.



Answer: Since the vector field $\vec{F}$ is conservative, and the curves are simple closed curves, total circulation of $\vec{F}$ along the smiley is zero.