MAT 16A Max time: 1 hr 30 mins ~~**Final Exam**~~

Type your name here:____

Signature:____August 2, 2013

Before the exam begins:

- Fill in all boxes above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- check that you have all 8 pages of the exam. (The number of pages includes this cover page.)
- Write your name at the top (in the space provided) of **EACH** page. (This is to ensure your pages can be identified with you if the staple holding your exam stops working.)

During the exam:

- Keep your eyes on your own exam!
- No notes/books or electronics AT ALL!

Note that the exam length is exactly 1 hr 30 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	5	6	**Total**
Score							

Print your initials:

Trigonometry

- Formula:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b, \quad \sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\sin^2 x + \cos^2 x = 1$$
$$\sec^2 x - \tan^2 x = 1$$
$$\csc^2 x - \cot^2 x = 1.$$

- Particular values:

	$x=0$	$x=\frac{\pi}{6}$	$x=\frac{\pi}{4}$	$x=\frac{\pi}{3}$	$x=\frac{\pi}{2}$	$x=\pi$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	DNE	0
$\csc x$	DNE	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	DNE
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	DNE	-1
$\cot x$	DNE	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	DNE

Geometry

- *Circle:* If radius=r, then perimeter=$2\pi r$ and area=πr^2.
- *Sphere:* If radius=r, then surface area=$4\pi r^2$ and volume=$\frac{4}{3}\pi r^3$.
- *Solid:* Volume of a rectangular solid is length×width×height.
- *Cone:* If height=h and radius (of circular base)=r, then volume=$\frac{1}{3}\pi r^2 h$.

Logarithm

$$\log ab = \log a + \log b, \quad \text{for } a > 0, b > 0$$
$$\log \frac{a}{b} = \log a - \log b, \quad \text{for } a > 0, b > 0$$
$$\log a^b = b \log a, \quad \text{for } a > 0$$

$$e^{\ln a} = a, \quad \text{for } a > 0$$
$$\ln e^a = a, \quad \text{true for any real number } a$$
$$\ln e = 1$$
$$\ln 1 = 0.$$

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Problem 1: (10 points) Find the limit $\lim_{x \to -1} f(x)$ (if exists), where

$$f(x) = \begin{cases} \frac{1}{2}x^3 + 2, & \text{if } x \leq -1 \\ x^2 + 1, & \text{if } x > -1. \end{cases}$$

Solution: We compute

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \frac{1}{2}(-1)^3 + 2 \\ &= = -\frac{1}{2} + 2 \\ &= \frac{3}{2}, \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to -1^+} f(x) &= (-1)^2 + 1 \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

Since $\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$, the limit $\lim_{x \to -1} f(x)$ does not exist.

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Problem 2: (10 points) Find the equation of the tangent line to the graph of $f(x) = x^2 \sin x$ at $(0, 0)$.

Solution: Slope of the tangent line is $f'(0)$. Differentiating $f(x)$ with respect to x we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x^2] \sin x + x^2 \frac{d}{dx}[\sin x] \\ &= 2x \sin x + x^2 \cos x. \end{aligned}$$

Therefore $f'(0) = 0$. Consequently, equation of the tangent line is

$$\begin{aligned} & y - 0 = f'(0)(x - 0) \\ \textit{i.e.,} \qquad & y = 0. \end{aligned}$$

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Problem 3: (10 points) Find the derivative of the function $f(x) = x \tan e^{x^2}$.

Solution: Using product rule and chain rule we have

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[x] \tan e^{x^2} + x \frac{d}{dx}[\tan e^{x^2}] \\
&= \tan e^{x^2} + x \sec^2 e^{x^2} \frac{d}{dx}[e^{x^2}] \\
&= \tan e^{x^2} + [x \sec^2 e^{x^2}] e^{x^2} \frac{d}{dx}[x^2] \\
&= \tan e^{x^2} + [x \sec^2 e^{x^2}] e^{x^2} 2x \\
&= \tan e^{x^2} + 2x^2 e^{x^2} \sec^2 e^{x^2}.
\end{aligned}$$

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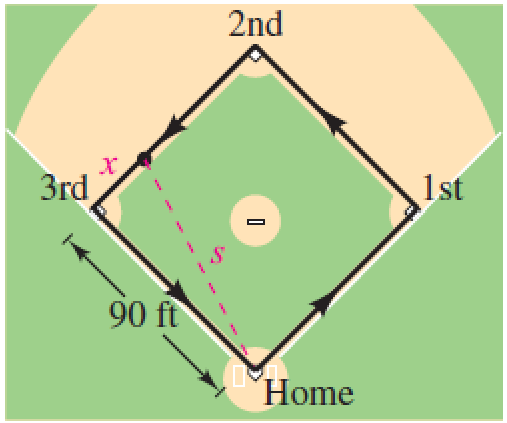
Problem 4: (10 points) A (square) baseball diamond has sides that are 90 feet long. A player 26 feet from third base is running at a speed of 30 feet per second. At what rate is the player's distance from home plate changing?



You don't need to simplify your answer.

Solution: Let player's distance from the home plate be s and from the third base is x (see figure). Therefore by Pythagorean theorem we have

$$s^2 = x^2 + 90^2. \tag{1}$$

We know that the player is running at a speed of 30 feet per second towards the third base i.e., $\frac{dx}{dt} = 30$ feet/sec. We want to find the rate of change of player's distance from the home plate i.e., $\frac{ds}{dt}$. Differentiating both sides of (1) with respect to t we have

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\textit{i.e.,} \qquad \frac{ds}{dt} = \frac{2x}{2s}\frac{dx}{dt}$$

$$\textit{i.e.,} \qquad \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}.$$

We know that $\frac{dx}{dt} = 30$ feet/sec. We know that $s^2 = x^2 + 90^2$. Therefore when $x = 26$ feet we have $s = \sqrt{26^2 + 90^2}$ feet. Hence value of $\frac{ds}{dt}$ when $x = 26$ feet is

$$\frac{ds}{dt} = \frac{26 \times 30}{\sqrt{26^2 + 90^2}} \text{ feet/sec.}$$

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Problem 5: (10 points) Find all relative extrema of the function $f(x) = 2x^3 - 15x^2 + 36x$.

Solution: Differentiating the function with respect to x we obtain

$$f'(x) = 6x^2 - 30x + 36.$$

To find the critical points we solve

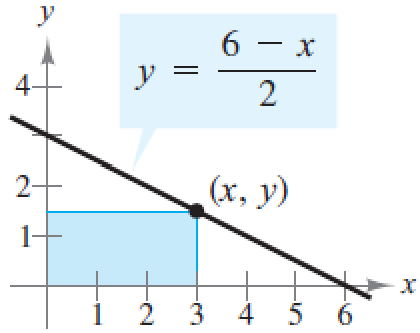
$$\begin{aligned} & f'(x) = 0 \\ i.e., \quad & 6x^2 - 30x + 36 = 0 \\ i.e., \quad & x^2 - 5x + 6 = 0 \\ i.e., \quad & (x-2)(x-3) = 0 \\ i.e., \quad & x = 2, 3. \end{aligned}$$

Now we do the following test

Test intervals	$-\infty < x < 2$	$2 < x < 3$	$3 < x < \infty$
Test points	$x = 0$	$x = \frac{5}{2}$	$x = 4$
Sign of $f'(x)$	$f'(0) = 36 > 0$	$f'\left(\frac{5}{2}\right) = -\frac{3}{2} < 0$	$f'(x) = 12 > 0$
Conclusion	Increasing	Decreasing	Increasing

Print your initials:

Problem 6: (10 points) A rectangle is bounded by the x and y axes and the graph of $y = \frac{6-x}{2}$. What length and width should the rectangle have so that its area is maximum?



Solution: Let (x, y) be the coordinate of the top-right corner of the rectangle. Notice that length and width of the rectangle are respectively x and y. Therefore area of the rectangle is

$$A = xy.$$

But we know that $y = \frac{6-x}{2}$. Therefore

$$\begin{aligned} A &= x \cdot \frac{6-x}{2} \\ &= \frac{1}{2}(6x - x^2). \end{aligned}$$

Differentiating A with respect to x we have

$$\begin{aligned} \frac{dA}{dx} &= \frac{1}{2}(6 - 2x) \\ &= 3 - x, \end{aligned}$$

and

$$\frac{d^2A}{dx^2} = -1.$$

To find the critical numbers we solve

$$\begin{aligned} & \frac{dA}{dx} = 0 \\ i.e., \quad & 3 - x = 0 \\ i.e., \quad & x = 3. \end{aligned}$$

Now we test

$$\left.\frac{d^2A}{dx^2}\right|_{x=3} = -1 < 0$$

Therefore at $x = 3$ the area A will be maximized. Length of the maximum rectangle is $x = 3$ unit and width is $y = \frac{6-3}{2} = \frac{3}{2}$ unit.