Section 3.4

[2] Let x and y be two positive numbers. We know that $x+y=S$. We want to maximize $P := xy$. Since $x+y=S$, therefore $y=S-x$. Hence we have

$$\begin{aligned} P &= x(S-x) \\ &= Sx - x^2 \end{aligned}$$

Differentiating P with respect x we have

$$\frac{dP}{dx} = S - 2x \quad \text{(since } S \text{ is a constant)}$$

and

$$\frac{d^2P}{dx^2} = -2.$$

Solving the equation $\frac{dP}{dx} = 0$ we obtain the critical number $x = \frac{S}{2}$. Notice that

$$\left.\frac{d^2P}{dx^2}\right|_{x=\frac{S}{2}} = -2 < 0.$$

Therefore by second derivative test, $x = \frac{S}{2}$ is a maxima. Hence the maximum possible value of P is

$$\begin{aligned} P_{max} &= S \cdot \frac{S}{2} - \left(\frac{S}{2}\right)^2 \\ &= \frac{S^2}{2} - \frac{S^2}{4} \\ &= \frac{S^2}{4}. \end{aligned}$$

[7] We want to minimize $f(x) = x + \frac{1}{x}$. Differentiating f twice with respect to x we obtain

$$f'(x) = 1 - \frac{1}{x^2},$$

and

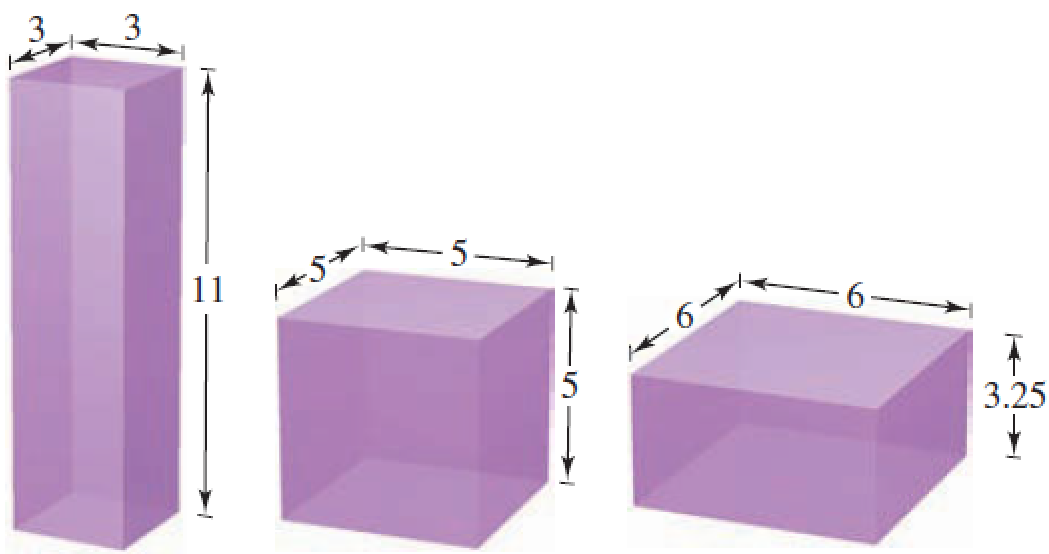
$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[1 - \frac{1}{x^2}\right] \\ &= \frac{d}{dx}[1 - x^{-2}] \\ &= -(-2)x^{-3} \\ &= \frac{2}{x^3}. \end{aligned}$$

Now to find the critical numbers we solve

$$\begin{aligned} & f'(x) = 0 \\ i.e., \quad & 1 - \frac{1}{x^2} = 0 \\ i.e., \quad & 1 = \frac{1}{x^2} \\ i.e., \quad & x^2 = 1 \\ i.e., \quad & x = \pm 1. \end{aligned}$$

But we want to restrict our attention to positive numbers only. So we will only consider $x = 1$. Now we notice that $f''(1) = 2 > 0$ Therefore $x = 1$ is a minima, and $x = 1$ is the positive number which minimizes $f(x)$.

[15]



(a) Surface area of top and bottom surfaces of the first box is $3 \times 3 = 9$ sq in. Surface of the other surfaces of the first box is $3 \times 11 = 33$ sq in. therefore total surface area of the first box is $(2 \times 9) + (4 \times 33) = 18 + 132 = 150$ sq in.

Similarly, total surface area of the second box is $6 \times (5 \times 5) = 6 \times 25 = 150$ sq in.

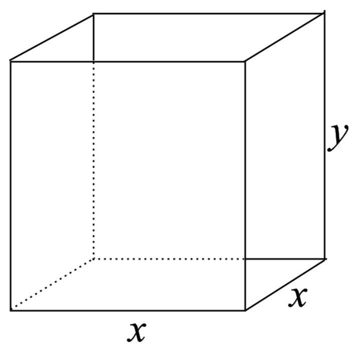
Total surface area of the third box is $[2 \times (6 \times 6)] + [4 \times (6 \times 3.25)] = 72 + 78 = 150$ sq in.

(b) Volume of the first box is $(3 \times 3) \times 11 = 99$ cubic in.

Volume of the second box is $5 \times 5 \times 5 = 125$ cubic in.

Volume of the third box is $6 \times 6 \times 3.25 = 117$ cubic in.

(c) Let side length of square base is x in and height of the box is y in. Then volume of the box is $V = x^2 y$. We want to maximize V.



Notice that surface area of the box is $2x^2 + 4xy$. Since total surface area of the box is 150 square inch, we have $2x^2 + 4xy = 150$. Solving y in terms of x we have $y = \frac{150-2x^2}{4x}$. Substituting this y in $V = x^2 y$ we have

$$\begin{aligned} V &= x^2 \frac{150 - 2x^2}{4x} \\ &= x \frac{150 - 2x^2}{4} \\ &= x \frac{75 - x^2}{2} \\ &= \frac{1}{2}(75x - x^3). \end{aligned}$$

Differentiating V twice with respect to x we obtain

$$\frac{dV}{dx} = \frac{1}{2}(75 - 3x^2),$$

and

$$\begin{aligned} \frac{d^2V}{dx^2} &= -\frac{3}{2} \cdot 2x \\ &= -3x. \end{aligned}$$

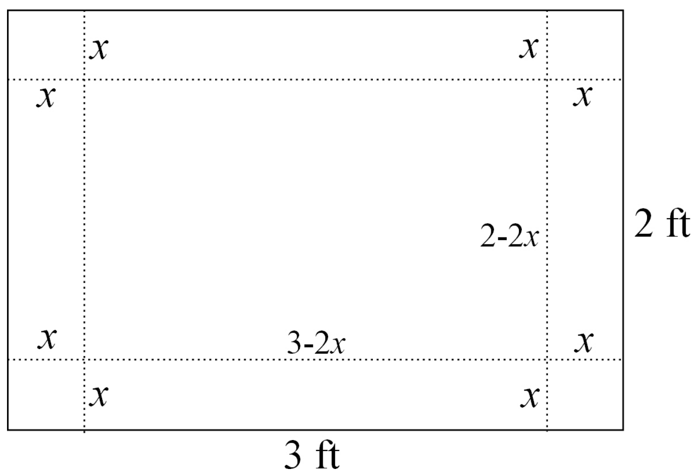
To find the critical numbers we solve $\frac{dV}{dx} = 0$

$$\begin{aligned} & \frac{1}{2}(75 - 3x^2) = 0 \\ i.e., \quad & 75 - 3x^2 = 0 \\ i.e., \quad & 3x^2 = 75 \\ i.e., \quad & x^2 = 25 \\ i.e., \quad & x = \pm 5. \end{aligned}$$

But it is not possible to have a negative side length. Therefore only possible value is $x = 5$. Now we see that $\left.\frac{d^2V}{dx^2}\right|_{x=5} = -15 < 0$. Therefore $x = 5$ is a maxima. If $x = 5$, we have $y = \frac{150-(2\times 25)}{4\times 5} = \frac{150-50}{20} = 5$.

Therefore dimension of the maximum sized box is 5 $in\times$5 $in\times$ 5in (length×width×height).

[19]



This problem is similar to the problem 18. Let side length of the squares cut from the corners is x ft. Then volume of the open top box is $V = (3-2x)(2-2x)x = 2(3-2x)(1-x)x$. Now differentiating V with respect to x twice we have

$$\begin{aligned} \frac{dV}{dx} &= 2(-2)(1-x)x + 2(-1)(3-2x)x + 2(3-2x)(1-x) \quad \text{(product rule)} \\ &= -4x(1-x) - 2x(3-2x) + 2(3-2x)(1-x) \\ &= -4x + 4x^2 - 6x + 4x^2 + 6 - 10x + 4x^2 \\ &= 12x^2 - 20x + 6, \end{aligned}$$

and

$$\frac{d^2V}{dx^2} = 24x - 16.$$

To find critical points we solve

$$\begin{aligned}
& \frac{dV}{dx} = 0 \\
\text{i.e.,} \quad & 12x^2 - 20x + 6 = 0 \\
\text{i.e.,} \quad & 6x^2 - 10x + 3 = 0 \\
\text{i.e.,} \quad & x = \frac{10 \pm \sqrt{(-10)^2 - (4 \times 6 \times 3)}}{2 \times 6} \qquad (\dagger) \\
\text{i.e.,} \quad & x = \frac{10 \pm \sqrt{100 - 72}}{12} \\
\text{i.e.,} \quad & x = \frac{10 \pm \sqrt{28}}{12} \\
\text{i.e.,} \quad & x = \frac{10 \pm \sqrt{4 \times 7}}{12} \\
\text{i.e.,} \quad & x = \frac{10 \pm \sqrt{4} \cdot \sqrt{7}}{12} \\
\text{i.e.,} \quad & x = \frac{10 \pm 2\sqrt{7}}{12} \\
\text{i.e.,} \quad & x = \frac{5 \pm \sqrt{7}}{6}.
\end{aligned}$$

So there are two critical points $x = \frac{5+\sqrt{7}}{6}$ and $x = \frac{5-\sqrt{7}}{6}$. Now we apply second derivative test. We notice that

$$\begin{aligned}
f''\left(\frac{5+\sqrt{7}}{6}\right) &= 24 \cdot \frac{5+\sqrt{7}}{6} - 20 \\
&= 4(5+\sqrt{7}) - 20 \\
&= 20 + 4\sqrt{7} - 20 \\
&= 4\sqrt{7} > 0,
\end{aligned}$$

and similarly

$$\begin{aligned}
f''\left(\frac{5-\sqrt{7}}{6}\right) &= 24 \cdot \frac{5-\sqrt{7}}{6} - 20 \\
&= -4\sqrt{7} < 0.
\end{aligned}$$

Therefore by second derivative test, $x = \frac{5-\sqrt{7}}{6}$ is a maxima. Consequently, the largest possible volume is

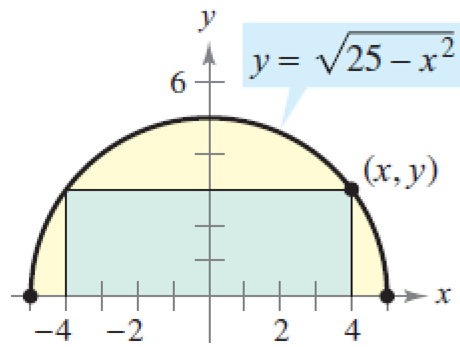
$$
\begin{aligned}
V_{max} &= 2\left[3 - 2\cdot\frac{5-\sqrt{7}}{6}\right]\left[1-\frac{5-\sqrt{7}}{6}\right]\cdot\frac{5-\sqrt{7}}{6} \\
&= 2\left[3-\frac{5-\sqrt{7}}{3}\right]\left[\frac{6-5+\sqrt{7}}{6}\right]\cdot\frac{5-\sqrt{7}}{6} \\
&= 2\left[\frac{9-5+\sqrt{7}}{3}\right]\left[\frac{6-5+\sqrt{7}}{6}\right]\cdot\frac{5-\sqrt{7}}{6} \\
&= 2\cdot\frac{4+\sqrt{7}}{3}\cdot\frac{1+\sqrt{7}}{6}\cdot\frac{5-\sqrt{7}}{6} \\
&= 2\frac{(4+\sqrt{7})(1+\sqrt{7})(5-\sqrt{7})}{3\times 6\times 6} \\
&= \frac{(4+5\sqrt{7}+7)(5-\sqrt{7})}{3\times 3\times 6} \\
&= \frac{(11+5\sqrt{7})(5-\sqrt{7})}{54} \\
&= \frac{55-11\sqrt{7}+25\sqrt{7}-(5\times 7)}{54} \\
&= \frac{55+14\sqrt{7}-35}{54} \\
&= \frac{20+14\sqrt{7}}{54} \\
&= \frac{10+7\sqrt{7}}{27}\text{ cubic ft.}
\end{aligned}
$$

(†): Here we are using general formula of solving a quadratic equation. The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

[26] Let (x, y) be the coordinate of the top-right corner of the rectangle. Then length of the rectangle is $2x$ and width/height of the rectangle is y. Consequently, area of the rectangle is $A = 2xy$.

But we know that the point (x, y) lies on the semicircle $y = \sqrt{25 - x^2}$. So we have the area of the rectangle $A = 2x\sqrt{25 - x^2}$. We want to maximize A. Differentiating A twice



with respect to x we obtain

$$\begin{aligned}
\frac{dA}{dx} &= \frac{d}{dx}[2x\sqrt{25-x^2}] \\
&= \frac{d}{dx}[2x]\sqrt{25-x^2} + 2x\frac{d}{dx}[\sqrt{25-x^2}] \\
&= 2\sqrt{25-x^2} + 2x\left(\frac{1}{2}(25-x^2)^{\frac{1}{2}-1}\right)\frac{d}{dx}[25-x^2] \\
&= 2\sqrt{25-x^2} + x(25-x^2)^{-\frac{1}{2}}(-2x) \\
&= 2\sqrt{25-x^2} - \frac{2x^2}{\sqrt{25-x^2}} \\
&= \frac{2(\sqrt{25-x^2}\cdot\sqrt{25-x^2}) - 2x^2}{\sqrt{25-x^2}} \\
&= \frac{2(25-x^2) - 2x^2}{\sqrt{25-x^2}} \\
&= \frac{50 - 2x^2 - 2x^2}{\sqrt{25-x^2}} \\
&= \frac{50 - 4x^2}{\sqrt{25-x^2}},
\end{aligned}$$

and

$$
\begin{aligned}
\frac{d^2A}{dx^2} &= \frac{\sqrt{25-x^2}\frac{d}{dx}[50-4x^2]-(50-4x^2)\frac{d}{dx}[\sqrt{25-x^2}]}{(\sqrt{25-x^2})^2} \\
&= \frac{-8x\sqrt{25-x^2}-(50-4x^2)\left[-\frac{x}{\sqrt{25-x^2}}\right]}{25-x^2} \\
&= \frac{-8x\sqrt{25-x^2}+\frac{x(50-4x^2)}{\sqrt{25-x^2}}}{25-x^2} \\
&= \frac{-8x(25-x^2)+x(50-4x^2)}{(25-x^2)\sqrt{25-x^2}} \quad \text{(multiplying numerator} \\
& \quad \text{and denominator by } \sqrt{25-x^2}) \\
&= \frac{-200x+8x^3+50x-4x^3}{(25-x^2)^{\frac{3}{2}}} \\
&= \frac{4x^3-150x}{(25-x^2)^{\frac{3}{2}}} \\
&= \frac{2x(2x^2-75)}{(25-x^2)^{\frac{3}{2}}}.
\end{aligned}
$$

To find the critical points we solve

$$
\begin{aligned}
& \frac{dA}{dx} = 0 \\
i.e., \quad & \frac{50-4x^2}{\sqrt{25-x^2}} = 0 \\
i.e., \quad & 50-4x^2 = 0 \\
i.e., \quad & 4x^2 = 50 \\
i.e., \quad & x^2 = \frac{50}{4} = \frac{25}{2} \\
i.e., \quad & x = \pm\sqrt{\frac{25}{2}} = \pm\frac{5}{\sqrt{2}}.
\end{aligned}
$$

Since (x, y) is the coordinate of the top-right corner of the rectangle, x cannot be negative. Therefore the only critical point is $x = \frac{5}{\sqrt{2}}$. We compute

$$
\begin{aligned}
f''\left(\frac{5}{\sqrt{2}}\right) &= \frac{2\cdot\frac{5}{\sqrt{2}}\left(2\cdot\frac{25}{2}-75\right)}{\left(25-\frac{25}{2}\right)^{\frac{3}{2}}} \\
&= \frac{5\sqrt{2}\cdot(-50)}{\left(\frac{25}{2}\right)^{\frac{3}{3}}} < 0.
\end{aligned}
$$

Therefore $x = \frac{5}{\sqrt{2}}$ is a maxima. We know that the length and width of the rectangle are respectively $2x$ and $y = \sqrt{25-x^2}$. Consequently, the optimal length is $2 \times \frac{5}{\sqrt{2}} = 5\sqrt{2}$ and the width is $\sqrt{25 - \frac{25}{2}} = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$.

Remark: Note that $x = \pm 5$ are also critical points. Because $\frac{dA}{dx}$ is undefined at $x = \pm 5$ and A is well defined at $x = \pm 5$.
But we don't need to worry about those critical points. Because at $x = \pm 5$, the area of the rectangle is $A = \pm 10\sqrt{25 - 25} = 0$. But we want to maximize the area of the rectangle and obviously $A = 0$ can not be the maximum area.