Section 3.3

[1] Given function $f(x) = x^2 - x - 2$. Differentiating twice with respect to $x$ we get $f'(x) = 2x - 1$ and $f''(x) = 2$. Since $f''(x) > 0$ for all $x$, the given function is concave upward.

[4] Given function $f(x) = \frac{x^2+4}{4-x^2}$. Differentiating with respect to $x$ we get

$$\begin{aligned} f'(x) &= \frac{(4-x^2)\frac{d}{dx}[x^2+4] - (x^2+4)\frac{d}{dx}[4-x^2]}{(4-x^2)^2} \\ &= \frac{2x(4-x^2)+2x(x^2+4)}{(4-x^2)^2} \\ &= \frac{8x-2x^2+2x^2+8x}{(4-x^2)^2} \\ &= \frac{16x}{(4-x^2)^2}, \end{aligned}$$

and

$$\begin{aligned} f''(x) &= \frac{(4-x^2)^2\frac{d}{dx}[16x] - 16x\frac{d}{dx}[(4-x^2)^2]}{(4-x^2)^4} \\ &= \frac{16(4-x^2)^2 - 16x[2(4-x^2)]\frac{d}{dx}[(4-x^2)]}{(4-x^2)^4} \\ &= \frac{16(4-x^2)^2 + 64x^2(4-x^2)}{(4-x^2)^4} \\ &= \frac{16(4-x^2)+64x^2}{(4-x^2)^3} \quad \text{(dividing both numerator and denominator by } (4-x^2)) \\ &= \frac{64-16x^2+64x^2}{(4-x^2)^3} \\ &= \frac{64+48x^2}{(4-x^2)^3} \\ &= \frac{16(4+3x^2)}{(4-x^2)^3}. \end{aligned}$$

Notice that $f''(x) = 0$ has no solution, but $f''(x)$ is undefined when $4 - x^2 = 0$ i.e., $x = \pm 2$. We have the following

| Test Interval | $-\infty < x < -2$ | $-2 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Test value | $x = -3$ | $x = 0$ | $x = 3$ |
| Sign of $f''(x)$ | $f''(-3) < 0$ | $f''(0) = 1 > 0$ | $f''(3) < 0$ |
| Conclusion | Concave downward | Concave upward | Concave downward |

[11] Given function $f(x) = x^3 - 5x^2 + 7x$. Differentiating twice with respect to $x$ we obtain $f'(x) = 3x^2 - 10x + 7$ and $f''(x) = 6x - 10$. To find critical points we solve $f'(x) = 0$.

$$\begin{aligned} & 3x^2 - 10x + 7 = 0 \\ i.e., \quad & 3x^2 - 3x - 7x + 7 = 0 \\ i.e., \quad & 3x(x-1) - 7(x-1) = 0 \\ i.e., \quad & (3x-7)(x-1) = 0 \\ i.e., \quad & x = \frac{7}{3}, 1. \end{aligned}$$

We compute $f''\left(\frac{7}{3}\right) = 6 \cdot \frac{7}{3} - 10 = 14 - 10 = 4 > 0$. Therefore $x = \frac{7}{3}$ is a relative minima. On the other hand $f''(1) = 6 - 10 = -4 < 0$. Therefore $x = 1$ is a relative maxima.

[16] Given function $f(x) = \sqrt{4 - x^2}$. Differentiating twice with respect to $x$ we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sqrt{4-x^2}] \\ &= \frac{d}{dx}\left[(4-x^2)^{\frac{1}{2}}\right] \\ &= \frac{1}{2}(4-x^2)^{\frac{1}{2}-1}\frac{d}{dx}[4-x^2] \\ &= \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) \\ &= -\frac{x}{\sqrt{4-x^2}}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} f''(x) &= -\frac{\sqrt{4-x^2}\frac{d}{dx}[x] - x\frac{d}{dx}[\sqrt{4-x^2}]}{4-x^2} \\ &= -\frac{\sqrt{4-x^2} - x\left[-\frac{x}{\sqrt{4-x^2}}\right]}{4-x^2} \quad \text{(using the equation (1))} \\ &= -\frac{\sqrt{4-x^2} + \left[\frac{x^2}{\sqrt{4-x^2}}\right]}{4-x^2} \\ &= -\frac{(4-x^2)+x^2}{(4-x^2)^{\frac{3}{2}}} \quad \text{(multiplying both numerator and denominator by } \sqrt{4-x^2}) \\ &= -\frac{4}{(4-x^2)^{\frac{3}{2}}}. \end{aligned}$$

Now solving the equation $f'(x) = 0$ we obtain $x = 0$. Also we notice that $f'(x)$ is undefined for $4 - x^2 = 0$ i.e., $x = \pm 2$, but $f(\pm 2) = 0$ i.e., $f(x)$ is well defined for $x = \pm 2$. Therefore we have three critical points $x = 0, \pm 2$.

Now we want to use second derivative test. Compute $f''(0) = -\frac{1}{2} > 0$, therefore $x = 0$ is a relative maxima.

But notice that $f''(-2)$ and $f''(2)$ are undefined. So we can not use the second derivative test for the critical points $x = \pm 2$. We have to use first derivative test for $x = \pm 2$.

| Test Interval | $-2 < x < 0$ | $0 < x < 2$ |
|---|---|---|
| Test value | $x = -1$ | $x = 1$ |
| Sign of $f'(x)$ | $f'(-1) = \frac{1}{\sqrt{3}} > 0$ | $f'(1) = -\frac{1}{\sqrt{3}} < 0$ |
| Conclusion | Increasing | decreasing |

[Note: We have excluded the test intervals $-\infty < x < -2$ and $2 < x < \infty$ because the function $f(x) = \sqrt{4 - x^2}$ is not defined in those intervals. Notice that $\sqrt{4 - x^2}$ is undefined for $4 - x^2 < 0$ i.e., $x > 2$ and $x < -2$].

Since we have no information on the intervals $-\infty < x < -2$ and $2 < x < \infty$, we can not apply the first derivative test, and we have no conclusion about $x = \pm 2$.

[21] Given function $f(x) = 5 + 3x^2 - x^3$. Differentiating with respect to $x$ we have $f'(x) = 6x - 3x^2$ and $f''(x) = 6 - 6x$. To find critical points we solve $f'(x) = 0$. Which gives us

$$\begin{aligned} & 6x - 3x^2 = 0 \\ i.e., \quad & 3x(2 - x) = 0 \\ i.e., \quad & x = 0, 2. \end{aligned}$$

Notice that $f''(0) = 6 > 0$ and $f''() = 6 - 12 = -6 < 0$. Therefore by second derivative test, $x = 0$ is a relative minima and $x = 2$ is a relative maxima.

[23] Since the function is increasing, $f'(x) > 0$ on the interval $(0, 2)$. We also notice that the function is concave upward on the interval $(0, 2)$. Therefore $f''(x) > 0$ on the interval $(0, 2)$.

[26] The function is decreasing on the interval $(0, 2)$. Therefore $f'(x) < 0$ on $(0, 2)$. Since the graph is concave upward, $f''(x) > 0$ on the interval $(0, 2)$.

[30] Given function $f(x) = x^4 - 18x^2 + 5$. Differentiating twice with respect to $x$ we have $f'(x) = 4x^3 - 36x$ and $f''(x) = 12x^2 - 36 = 12(x^2 - 3)$. To determine the test intervals we solve the equation $f''(x) = 0$

$$\begin{aligned} & 12(x^2 - 3) = 0 \\ i.e., \quad & x^2 - 3 = 0 \\ i.e., \quad & x^2 = 3 \\ i.e., \quad & x = \pm\sqrt{3}. \end{aligned}$$

Notice that $f''(x)$ is well defined everywhere. Therefore we have the following

| Test Intervals | $-\infty < x < -\sqrt{3}$ | $-\sqrt{3} < x < \sqrt{3}$ | $\sqrt{3} < x < \infty$ |
|---|---|---|---|
| Test Values | $x = -4$ | $x = 0$ | $x = 4$ |
| Sign of $f''(x)$ | $f''(-4) = 156 > 0$ | $f''(0) = -36 < 0$ | $f''(4) = 156 > 0$ |
| Conclusion | Concave upward | Concave downward | Concave upward |

We notice that concavity of the given function changes at $x = -\sqrt{3}$ (function is concave upward on the left side of $x = -\sqrt{3}$ and concave downward on the right hand side of $x = -\sqrt{3}$). Therefore $x = -\sqrt{3}$ is a point of inflection. For the similar reason, $x = \sqrt{3}$ is also a point of inflection.

**Quiz 5:** The quiz problem was $f(x) = x^4 - 24x^2 + 7$. This is similar as problem 30. The answer of the quiz problem is $x = \pm 2$.

[33] Given function $h(x) = (x-2)^3(x-1)$. Differentiating twice we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}[(x-2)^3](x-1) + (x-2)^3\frac{d}{dx}[(x-1)] \\ &= 3(x-2)^2(x-1) + (x-2)^3 \\ &= (x-2)^2[3(x-1) + (x-2)] \\ &= (x-2)^2(4x-5), \end{aligned}$$

and

$$\begin{aligned} f''(x) &= \frac{d}{dx}[(x-2)^2](4x-5) + (x-2)^2\frac{d}{dx}[(4x-5)] \\ &= 2(x-2)(4x-5) + 4(x-2)^2 \\ &= 2(x-2)[(4x-5) + 2(x-2)] \\ &= 2(x-2)(6x-9). \end{aligned}$$

To find the possible points of inflection we solve the equation $f''(x) = 0$. Solving $f''(x) = 0$, we obtain $x = 2$ and $x = \frac{9}{6} = \frac{3}{2}$. Since $f''(x)$ is not undefined anywhere, we have

| Test Intervals | $-\infty < x < \frac{3}{2}$ | $\frac{3}{2} < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Test Values | $x = 0$ | $x = \frac{7}{4}$ | $x = 3$ |
| Sign of $f''(x)$ | $f''(0) = 36 > 0$ | $f''\left(\frac{7}{4}\right) = -\frac{3}{2} < 0$ | $f''(3) = 18 > 0$ |
| Conclusion | Concave upward | Concave downward | Concave upward |

Since concavity of the function changes at $x = \frac{3}{2}$ and $x = 2$, both $x = \frac{3}{2}$ and $x = 2$ are points of inflection.