
Section 3.2

[1] Given function $f(x) = -2x^2 + 4x + 3$. Differentiating with respect to x we obtain

$$f'(x) = -4x + 4.$$

Solving $f'(x) = 0$ we get $x = 1$. $f'(x)$ is well defined for all real numbers. Therefore the only critical point is $x = 1$.

Test intervals	$-\infty < x < 1$	$1 < x < \infty$
Test points	$x = 0$	$x = 2$
Sign of $f'(x)$	$f'(0) = 4 > 0$	$f'(2) = -4 < 0$
Conclusion	Increasing	Decreasing

By first derivative test, $x = 1$ is a relative maxima.

[4] $f(x) = -4x^2 + 4x + 1$. Differentiating with respect to x we obtain $f'(x) = -8x + 4$. Solving $f'(x) = 0$ we have $x = \frac{1}{2}$. Since $f'(x)$ is well defined for all real numbers, the only critical point is $x = \frac{1}{2}$.

Test intervals	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Test points	$x = 0$	$x = 1$
Sign of $f'(x)$	$f'(0) = 4 > 0$	$f'(1) = -4 < 0$
Conclusion	Increasing	Decreasing

By first derivative test, $x = \frac{1}{2}$ is a relative maxima.

[7] $h'(x) = -(x + 4)^3$. Differentiating with respect to x we have $h'(x) = -3(x + 4)^2$. Solving $f'(x) = 0$ we get $x = -4$. Since $f'(x)$ is well defined for all real numbers, there is only one critical point $x = -4$.

Test intervals	$-\infty < x < -4$	$-4 < x < \infty$
Test points	$x = -5$	$x = -3$
Sign of $f'(x)$	$f'(-5) = -3 < 0$	$f'(0) = -3 < 0$
Conclusion	Decreasing	Decreasing

Since $f'(x)$ has same sign on left and right side of $x = -4$, therefore $x = -4$ is neither a relative maxima nor a relative minima.

[12] $f(x) = x^4 - 12x^3$. Differentiating with respect to x we get $f'(x) = 4x^3 - 36x^2$. To solve $f'(x) = 0$

$$\begin{aligned} & 4x^3 - 36x^2 = 0 \\ i.e., \quad & x^3 - 9x^2 = 0 \\ i.e., \quad & x^2(x-9) = 0 \\ i.e., \quad & x = 0, 9. \end{aligned}$$

Since $f'(x)$ is well defined for all real numbers, $x = 0, 9$ are the only critical points.

Test intervals	$-\infty < x < 0$	$0 < x < 9$	$9 < x < \infty$
Test points	$x = -1$	$x = 1$	$x = 10$
Sign of $f'(x)$	$f'(-1) = -40 < 0$	$f'(1) = -32 < 0$	$f'(10) = 400 > 0$
Conclusion	Decreasing	Decreasing	Increasing

By first derivative test, $x = 0$ is nether a relative maxima nor a relative minima, and $x = 9$ is a relative minima.

[16] $f(x) = x + \frac{1}{x}$. Differentiating with respect to x we have $f'(x) = 1 - \frac{1}{x^2}$. To solve $f'(x) = 0$

$$\begin{aligned} & 1 - \frac{1}{x^2} = 0 \\ i.e., \quad & 1 = \frac{1}{x^2} \\ i.e., \quad & x^2 = 1 \\ i.e., \quad & x = \pm 1. \end{aligned}$$

Also notice that $f'(x)$ is undefined for $x = 0$. But $f(x)$ is also undefined for $x = 0$. Therefore $x = 0$ is not a critical point, and we have two critical points $x = \pm 1$.

Test intervals	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test points	$x = -2$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-2) = \frac{3}{4} > 0$	$f'\left(\frac{1}{2}\right) = -3 < 0$	$f'(2) = \frac{3}{4} > 0$
Conclusion	Increasing	Decreasing	Increasing

By first derivative test, $x = -1$ is a relative maxima, and $x = 1$ is a relative minima.

Remark: The function $f(x) = x + \frac{1}{x}$ is undefined at $x = 0$. So more precisely, we should not consider the interval $-1 < x < 1$ as a test interval. Instead we should consider two intervals $-1 < x < 0$ and $0 < x < 1$ (to exclude $x = 0$ from our consideration). But that does not change the answer.

[18] $h(x) = \frac{4}{x^2+1}$. Differentiating with respect to x we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{4}{x^2+1}\right] \\ &= \frac{d}{dx}[4(x^2+1)^{-1}] \\ &= 4(-1)(x^2+1)^{-2}\frac{d}{dx}[x^2+1] \\ &= -4\frac{1}{(x^2+1)^2}\cdot 2x \\ &= \frac{-8x}{(x^2+1)^2}. \end{aligned}$$

Solving $f'(x) = 0$ we get $x = 0$. Since $x^2 + 1 \neq 0$ for all real numbers. Therefore $f'(x)$ is well defined for all real numbers. Hence the only critical point is $x = 0$.

Test intervals	$-\infty < x < 0$	$0 < x < \infty$
Test points	$x = -1$	$x = 1$
Sign of $h'(x)$	$f'(-1) = 2 > 0$	$f'(1) = -2 < 0$
Conclusion	Increasing	Decreasing

By first derivative test $x = 0$ is a relative maxima.

[25] Given function $h(s) = \frac{1}{3-s}$. Differentiating with respect s we have

$$\begin{aligned} h'(s) &= \frac{d}{ds}\left[\frac{1}{3-s}\right] \\ &= \frac{d}{ds}[(3-s)^{-1}] \\ &= -(3-s)^{-2}\frac{d}{ds}[3-s] \\ &= -(3-s)^{-2}[0-1] \\ &= (3-s)^{-2} \\ &= \frac{1}{(3-s)^2}. \end{aligned}$$

Note that there is no solution of $h'(s) = 0$. Also notice that both $h(s)$ and $h'(s)$ are undefined for $s = 3$. Therefore $s = 3$ is not a critical number. Consequently, there is no critical number (we don't have to check any relative maxima or minima).

We are looking at the function $h(s)$ restricted on the interval $[0, 2]$.

s value	End point: $s = 0$	End point: $s = 2$
$h(s)$	$h(0) = \frac{1}{3}$	$h(2) = 1$
Conclusion	Absolute Minimum	Absolute Maximum

[30] $g(x) = 4\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)$. Differentiating with respect to x we have

$$\begin{aligned} g'(x) &= 4\frac{d}{dx}[1 + x^{-1} + x^{-2}] \\ &= 4[-x^{-2} - 2x^{-3}] \\ &= -4\left[x^{-2} + 2x^{-3}\right] \\ &= -4\left[\frac{1}{x^2} + \frac{2}{x^3}\right] \\ &= -4\left[\frac{x+2}{x^3}\right]. \end{aligned}$$

Solving the equation $g'(x) = 0$ we have $x = -2$. Notice that both $g(x)$ and $g'(x)$ are undefined for $x = 0$. Therefore $x = 0$ is not a critical point, and we have only one critical point $x = -2$ inside the interval $[-4, 5]$.

x value	End point: $x = -4$	Critical point: $x = -2$	End point: $x = 5$
$g(x)$	$g(-4) = \frac{13}{4}$	$g(-2) = 3$	$g(5) = \frac{124}{25}$
Conclusion	None	Absolute Minimum	Absolute Maximum

[37] $f(x) = \frac{2x}{x^2+4}$. Differentiating with respect to x we have

$$\begin{aligned} f'(x) &= \frac{(x^2+4)\frac{d}{dx}[2x] - 2x\frac{d}{dx}[x^2+4]}{(x^2+4)^2} \\ &= \frac{2(x^2+4) - 4x^2}{(x^2+4)^2} \\ &= \frac{8 - 2x^2}{(x^2+4)^2}. \end{aligned}$$

To solve $f'(x) = 0$

$$\begin{aligned} & \frac{8 - 2x^2}{(x^2+4)^2} = 0 \\ \textit{i.e.,}\quad & 8 - 2x^2 = 0 \\ \textit{i.e.,}\quad & 4 - x^2 = 0 \\ \textit{i.e.,}\quad & x^2 = 4 \\ \textit{i.e.,}\quad & x = \pm 2. \end{aligned}$$

Notice that $f'(x)$ is well defined for all real numbers. Therefore there are only two critical points $x = \pm 2$.

x points	End point: $x = 0$	Critical number: $x = -2$	Critical number: $x = 2$	$\lim_{x\to\infty} f(x)$
$f(x)$	$f(0) = 0$	$f(-2) = -\frac{1}{2}$	$f(2) = \frac{1}{2}$	$0(\dagger)$
Conclusion	None	Absolute Minimum	Absolute Maximum	None

† **Clarification:** Why $\lim_{x\to\infty} f(x) = 0$?

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x}{x^2+4} &= \lim_{x\to\infty} \frac{\frac{2}{x}}{1+\frac{4}{x^2}} \quad \text{(dividing top and bottom by } x^2\text{)} \\
&= \frac{\lim_{x\to\infty}\left[\frac{2}{x}\right]}{\lim_{x\to\infty}\left[1+\frac{4}{x^2}\right]} \\
&= \frac{0}{1+0} \\
&= 0.
\end{aligned}$$