
Section 3.1

[1] $f(x) = \frac{x^2}{x^2+4}$. Differentiating with respect to x we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{x^2}{x^2+4}\right] \\ &= \frac{(x^2+4)\frac{d}{dx}[x^2] - x^2\frac{d}{dx}[x^2+4]}{(x^2+4)^2} \\ &= \frac{2x(x^2+4) - x^2[2x+0]}{(x^2+4)^2} \\ &= \frac{2x^3+8x-2x^3}{(x^2+4)^2} \\ &= \frac{8x}{(x^2+4)^2}. \end{aligned}$$

Given points are $(0,0)$, $\left(1, \frac{1}{5}\right)$, and $\left(-1, \frac{1}{5}\right)$. Values of $f'(x)$ at those points are

$$f'(0) = \frac{0}{(0+4)^2} = \frac{0}{16} = 0,$$

$$\begin{aligned} f'(1) &= \frac{8}{(1+4)^2} \\ &= \frac{8}{5^2} = \frac{8}{25}, \end{aligned}$$

and

$$\begin{aligned} f'(-1) &= \frac{-8}{(1+4)^2} \\ &= \frac{-8}{5^2} = -\frac{8}{25}. \end{aligned}$$

[4] Given function $f(x) = -3x\sqrt{x+1}$. Differentiating with respect to x we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}[-3x\sqrt{x+1}] \\ &= \frac{d}{dx}[-3x]\sqrt{x+1} - 3x\frac{d}{dx}[\sqrt{x+1}] \\ &= -3\sqrt{x+1} - 3x\frac{d}{dx}[(x+1)^{1/2}] \\ &= -3\sqrt{x+1} - 3x \times \frac{1}{2}(x+1)^{\frac{1}{2}-1}\frac{d}{dx}[x+1] \\ &= -3\sqrt{x+1} - \frac{3x}{2\sqrt{x+1}}. \end{aligned}$$

Given points are $(-1, 0)$, $\left(-\frac{2}{3}, \frac{2\sqrt{3}}{3}\right)$, and $(0, 0)$. Values of $f'(x)$ at those points are

$$\begin{aligned} f'(-1) &= 3\sqrt{-1+1} - \frac{3(-1)}{2\sqrt{-1+1}} \\ &= 3 \times 0 + \frac{3}{2 \times 0} \\ &= \frac{3}{0}. \end{aligned}$$

Since $\frac{3}{0}$ is undefined, $f'(0)$ is undefined.

$$\begin{aligned} f'\left(-\frac{2}{3}\right) &= -3\sqrt{-\frac{2}{3}+1} - \frac{3 \times \left(-\frac{2}{3}\right)}{2\sqrt{-\frac{2}{3}+1}} \\ &= -3\sqrt{\frac{1}{3}} + \frac{2}{2\sqrt{\frac{1}{3}}} \\ &= -3 \cdot \frac{1}{\sqrt{3}} + \frac{1}{\frac{1}{\sqrt{3}}} \\ &= -\sqrt{3} + \sqrt{3} \\ &= 0. \end{aligned}$$

$$\begin{aligned} f'(0) &= -3\sqrt{0+1} - \frac{0}{2\sqrt{0+1}} \\ &= -3 - \frac{0}{2} \\ &= -3. \end{aligned}$$

[7] $f(x) = x^4 - 2x^2$. First of all, we have to find the critical points. Differentiating $f(x)$ with respect to x we have

$$f'(x) = 4x^3 - 4x.$$

To find critical points we solve

$$\begin{aligned} & f'(x) = 0 \\ i.e., \quad & 4x^3 - 4x = 0 \\ i.e., \quad & x^3 - x = 0 \\ i.e., \quad & x(x^2 - 1) = 0 \\ i.e., \quad & x = 0, x^2 - 1 = 0 \\ i.e., \quad & x = 0, x^2 = 1 \\ i.e., \quad & x = 0, x = \pm 1. \end{aligned}$$

Also notice that $f'(x)$ is defined everywhere. Therefore $x = 0, \pm 1$. We do the following to determine the open intervals on which the function is increasing or decreasing

Test intervals	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Text points	$x = -2$	$x = -\frac{1}{2}$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-2) = -24 < 0$	$f'\left(-\frac{1}{2}\right) = \frac{3}{2} > 0$	$f'\left(\frac{1}{2}\right) = -\frac{3}{2} < 0$	$f'(2) = 24 > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

[8] Given function $f(x) = \frac{x^2}{x+1}$. Differentiating with respect to x we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{x^2}{x+1}\right] \\ &= \frac{(x+1)\frac{d}{dx}[x^2] - x^2\frac{d}{dx}[x+1]}{(x+1)^2} \\ &= \frac{2x(x+1) - x^2}{(x+1)^2} \\ &= \frac{2x^2 + 2x - x^2}{(x+1)^2} \\ &= \frac{x^2 + 2x}{(x+1)^2}. \end{aligned}$$

To solve the equation $f'(x) = 0$ we have

$$\begin{aligned} & \frac{x^2 + 2x}{(x+1)^2} = 0 \\ i.e., \quad & x^2 + 2x = 0 \\ i.e., \quad & x(x+2) = 0 \\ i.e., \quad & x = 0, -2. \end{aligned}$$

Notice that $f'(x)$ is undefined for $x = -1$, because $f'(-1) = \frac{1-2}{0} = \frac{-1}{0}$. But $f(x)$ is also undefined for $x = -1$. Therefore $x = -1$ is not a critical number, and the only critical numbers are $x = 0, -2$.

Now we proceed as following.

Test intervals	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Test points	$x = -3$	$x = -\frac{1}{2}$	$x = 1$
Sign of $f'(x)$	$f'(-3) = \frac{3}{4} > 0$	$f'\left(-\frac{1}{2}\right) = -3 < 0$	$f'(1) = \frac{3}{4} > 0$
Conclusion	Increasing	Decreasing	Increasing

[17] Given function $f(x) = \sqrt{x^2 - 1}$. Differentiating with respect to x we obtain

$$\begin{aligned} f'(x) &= \frac{d}{dx}[(x^2-1)^{1/2}] \\ &= \frac{1}{2}(x^2-1)^{\frac{1}{2}-1}\frac{d}{dx}[x^2-1] \\ &= \frac{1}{2}(x^2-1)^{-\frac{1}{2}} \cdot 2x \\ &= \frac{x}{\sqrt{x^2-1}}. \end{aligned}$$

Notice that there is no real solution of $f'(x) = 0$.

[Common mistake:

$$\begin{aligned} \frac{x}{\sqrt{x^2-1}} &= 0 \\ i.e., \quad x &= 0. \end{aligned}$$

But if we plugin $x = 0$ in $f'(x)$ we get $f'(0) = \frac{0}{\sqrt{-1}}$, which is not a real number. Because we have $\sqrt{\text{negative number}}$.]

Now we observe that $f'(x)$ is undefined if

$$\begin{aligned} & \sqrt{x^2-1} = 0 \\ i.e., \quad & x^2 - 1 = 0 \\ i.e., \quad & x^2 = 1 \\ i.e., \quad & x = \pm 1. \end{aligned}$$

We also notice that $f(\pm 1) = \sqrt{1-1} = 0$ i.e., $f(x)$ is well defined for $x = \pm 1$. Therefore $x = \pm 1$ are critical points. Ideally the test intervals should be $-\infty < x < -1$, $-1 < x < 1$, and $1 < x < \infty$. But the given function $f(x) = \sqrt{x^2-1}$ is undefined for $x^2 - 1 < 0$ (because negative number inside a square root is not allowed) i.e., for $-1 < x < 1$. So it does not make sense to talk about increasing or decreasing inside the interval $-1 < x < 1$. Therefore we have the following

Test intervals	$-\infty < x < -1$	$1 < x < \infty$
Test points	$x = -2$	$x = 2$
Sign of $f'(x)$	$f'(-2) = -\frac{2}{\sqrt{3}} < 0$	$f'(2) = \frac{2}{\sqrt{3}} > 0$
Conclusion	Decreasing	Increasing

[18] the given function $f(x) = \sqrt{4 - x^2}$. Differentiating with respect to x we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}[(4 - x^2)^{1/2}] \\ &= \frac{1}{2}(4 - x^2)^{\frac{1}{2}-1}\frac{d}{dx}[4 - x^2] \\ &= \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}[-2x] \\ &= -\frac{x}{\sqrt{4 - x^2}}. \end{aligned}$$

to solve the equation $f'(x) = 0$ we have

$$\begin{aligned} & -\frac{x}{\sqrt{4 - x^2}} = 0 \\ i.e., \quad & -x = 0 \\ i.e., \quad & x = 0. \end{aligned}$$

[Note that this problem is slightly different from the previous one. Plugging in $x = 0$ in $f'(x)$ we have $f'(0) = -\frac{0}{\sqrt{4}} = -\frac{0}{2} = 0$. Therefore $x = 0$ is indeed a solution of $f'(x) = 0$]. We also notice that $f'(x)$ is undefined for

$$\begin{aligned} & \sqrt{4 - x^2} = 0 \\ i.e., \quad & 4 - x^2 = 0 \\ i.e., \quad & 4 = x^2 \\ i.e., \quad & x = \pm 2. \end{aligned}$$

Whereas $f(\pm 2) = \sqrt{4 - 4} = 0$ i.e., $f(x)$ is well defined for $x = \pm 2$. Hence the critical points are $x = 0, \pm 2$. Ideally the test intervals should be $-\infty < x < -2$, $-2 < x < 0$, $0 < x < 2$, and $2 < x < \infty$. But observe that the given function $f(x) = \sqrt{4 - x^2}$ is undefined for $4 - x^2 < 0$ (because negative number inside a square root is not allowed) i.e., $4 < x^2$ i.e., $2 < x$ or $x < -2$. In other words the function is well defined if $4 - x^2 \geq 0$ i.e., if $x^2 \leq 4$ i.e., if $-2 \leq x \leq 2$. Therefore it makes sense to talk about increasing or decreasing only when $-2 \leq x \leq 2$. Consequently we have the following

Test intervals	$-2 < x < 0$	$0 < x < 2$
Test points	$x = -1$	$x = 1$
Sign of $f'(x)$	$f'(-1) = \frac{1}{\sqrt{3}} > 0$	$f'(1) = -\frac{1}{\sqrt{3}} < 0$
Conclusion	Increasing	Decreasing

[23] Given function $f(x) = x\sqrt{x+1}$. Differentiating with respect to x we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x\sqrt{x+1}] \\ &= \frac{d}{dx}[x]\sqrt{x+1} + x\frac{d}{dx}[\sqrt{x+1}] \\ &= \sqrt{x+1} + x\frac{1}{2}(x+1)^{-\frac{1}{2}}\frac{d}{dx}[x+1] \\ &= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}. \end{aligned}$$

To find the critical points we solve

$$\begin{aligned} & f'(x) = 0 \\ i.e., \quad & \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} = 0 \\ i.e., \quad & \sqrt{x+1} = -\frac{x}{2\sqrt{x+1}} \\ i.e., \quad & \sqrt{x+1} \cdot 2\sqrt{x+1} = -x \\ i.e., \quad & 2(x+1) = -x \\ i.e., \quad & 2x+2 = -x \\ i.e., \quad & 2x+2+x = 0 \\ i.e., \quad & 3x+2 = 0 \\ i.e., \quad & x = -\frac{2}{3}. \end{aligned}$$

We also notice that $f'(x)$ is undefined for $x = -1$ (because $f'(-1) = \sqrt{-1+1} + \frac{-1}{2\sqrt{-1+1}} = \frac{-1}{0}$). Whereas $f(-1) = (-1) \cdot \sqrt{-1+1} = (-1) \times 0 = 0$ i.e., $f(x)$ is well defined for $x = -1$. Therefore $x = -1$ is also a critical point. Consequently, we have two critical points $x = -\frac{2}{3}, -1$.

Ideally the test intervals should be $-\infty < x < -1$, $-1 < x < -\frac{2}{3}$, and $-\frac{2}{3} < x < \infty$. But notice that the given function $f(x) = x\sqrt{x+1}$ is undefined for $x+1 < 0$ (because $\sqrt{\text{negative number}}$ is not allowed) i.e., $x < -1$. Therefore we have the following

Test intervals	$-1 < x < -\frac{2}{3}$	$-\frac{2}{3} < x < \infty$
Test points	$x = -\frac{5}{6}$	$x = 0$
Sign of $f'(x)$	$f'\left(-\frac{5}{6}\right) = -\frac{3}{2\sqrt{6}} < 0$	$f'(0) = 1 > 0$
Conclusion	Decreasing	Increasing

Alternative method

We have $f'(x) = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$. We know that the function is increasing if $f'(x) > 0$

and decreasing if $f'(x) < 0$. Therefore the given function $f(x) = x\sqrt{x+1}$ is increasing if

$$\begin{aligned}
& f'(x) > 0 \\
i.e., \quad & \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} > 0 \\
i.e., \quad & \sqrt{x+1} > -\frac{x}{2\sqrt{x+1}} \\
i.e., \quad & 2(x+1) > -x \\
i.e., \quad & 2x + 2 + x > 0 \\
i.e., \quad & 3x + 2 > 0 \\
i.e., \quad & 3x > -2 \\
i.e., \quad & x > -\frac{2}{3}.
\end{aligned}$$

Therefore the function is increasing when $x > -\frac{2}{3}$. In other words, the function is increasing in the interval $-\frac{2}{3} < x < \infty$. Similarly the function is decreasing if

$$\begin{aligned}
& f'(x) < 0 \\
i.e., \quad & \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} < 0 \\
i.e., \quad & x < -\frac{2}{3}.
\end{aligned}$$

Therefore the function is decreasing in the interval $-\infty < x < -\frac{2}{3}$.

But we know that the function $f(x) = x\sqrt{x+1}$ is not defined for $x + 1 < 0$ i.e., $x < -1$. Therefore it does not make sense to talk about increasing or decreasing when $x < -1$. Consequently the function is decreasing in the interval $-1 < x < -\frac{2}{3}$ (not in the interval $-\infty < x < -\frac{2}{3}$).

Finally, the function is increasing in the interval $-\frac{2}{3} < x < \infty$ and decreasing in the interval $-1 < x < -\frac{2}{3}$.

Remark: This alternative method can be used for any problem about increasing or decreasing function. There are a few advantages in this alternative method. We don't have to compute test interval, test point, and sign of $f'(x)$ in each test interval.

[28] Given function $f(x) = \frac{x^2}{x^2+4}$. We compute

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{x^2}{x^2+4}\right] \\ &= \frac{(x^2+4)\frac{d}{dx}[x^2] - x^2\frac{d}{dx}[x^2+4]}{(x^2+4)^2} \\ &= \frac{2x(x^2+4) - x^2 \cdot 2x}{(x^2+4)^2} \\ &= \frac{2x^3+8x-2x^3}{(x^2+4)^2} \\ &= \frac{8x}{(x^2+4)^2} \end{aligned}$$

Solving the equation $f'(x) = 0$ we obtain $x = 0$. Notice that since always $x^2 + 4 \neq 0$, $f'(x)$ is defined everywhere. So, there is only one critical point, namely $x = 0$. Also $f(x)$ is defined everywhere. Consequently, we have the following

Test intervals	$-\infty < x < 0$	$0 < x < \infty$
Test points	$x = -1$	$x = 1$
Sign of $f'(x)$	$f'(-1) = -\frac{8}{25} < 0$	$f'(1) = \frac{8}{25} > 0$
Conclusion	Decreasing	Increasing

[33] Given function

$$f(x) = \begin{cases} 3x+1, & x \leq 1 \\ 5 - x^2, & x > 1. \end{cases}$$

Differentiating with respect to x we have

$$f'(x) = \begin{cases} 3, & x < 1 \quad \text{(notice, I have written } x < 1 \text{ not } x \leq 1\text{)} \\ -2x, & x > 1. \end{cases}$$

Solving $f'(x) = 0$ we obtain $x = 0$ when $x > 1$. But when x can not be 0 when $x > 1$. Therefore there is no solution of $f'(x) = 0$. On the other hand $f'(x)$ is undefined for $x = 1$ but $f(x)$ is well defined for $x = 1$. Therefore $x = 1$ is the only critical point.

Test intervals	$-\infty < x < 1$	$1 < x < \infty$
Test points	$x = 0$	$x = 2$
Sign of $f'(x)$	$f'(0) = 3 > 0$	$f'(2) = -4 < 0$
Conclusion	Increasing	Decreasing

Remark: If a function is defined piecewise, then take the joining points as critical points (like $x = 1$ in this problem)

Alternative method

We have

$$f'(x) = \begin{cases} 3, & x < 1 \\ -2x, & x > 1. \end{cases}$$

$$= \begin{cases} \text{positive}, & x < 1 \\ \text{negative}, & x > 1. \end{cases}$$

Therefore the function is increasing in the interval $-\infty < x < 1$ and decreasing in the interval $1 < x < \infty$.