Section 2.8

[1] Given equation $y = x^2 - \sqrt{x}$. Differentiating both sides with respect to t we have

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}[x^2 - \sqrt{x}] \\ &= 2x\frac{dx}{dt} - \frac{1}{2}x^{-1/2}\frac{dx}{dt} \\ &= \left[2x - \frac{1}{2\sqrt{x}}\right]\frac{dx}{dt}. \end{aligned} \tag{1}$$

(a) Plugging in $x = 4$ and $\frac{dx}{dt} = 8$ in the above equation (1) we get

$$\begin{aligned} \frac{dy}{dt} &= \left[8 - \frac{1}{2\sqrt{4}}\right] \times 8 \\ &= \left[8 - \frac{1}{4}\right] \times 8 \\ &= 64 - 2 = 62. \end{aligned}$$

(b) Plugging in $x = 16$ and $\frac{dy}{dt} = 12$ in (1) we obtain

$$\begin{aligned} & 12 = \left[32 - \frac{1}{2\sqrt{16}}\right]\frac{dx}{dt} \\ i.e., \quad & 12 = \left[32 - \frac{1}{8}\right]\frac{dx}{dt} \\ i.e., \quad & 12 = \frac{255}{8}\frac{dx}{dt} \\ i.e., \quad & \frac{dx}{dt} = \frac{96}{255} = \frac{32}{85}. \end{aligned}$$

[4] $x^2 + y^2 = 25$. Differentiating both sides with respect to t we get

$$\begin{aligned} & 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \\ i.e., \quad & x\frac{dx}{dt} + y\frac{dy}{dt} = 0. \end{aligned} \tag{2}$$

(a) Putting $x = 3, y = 4$, and $\frac{dx}{dt} = 8$ in (2) we get

$$
\begin{aligned}
& 24 + 4\frac{dy}{dt} = 0 \\
i.e., \quad & 4\frac{dy}{dt} = -24 \\
i.e., \quad & \frac{dy}{dt} = -6.
\end{aligned}
$$

(b) Putting $x = 4, y = 3$, and $\frac{dy}{dt} = -2$ in (2) we get

$$
\begin{aligned}
& 4\frac{dx}{dt} - 6 = 0 \\
i.e., \quad & 4\frac{dx}{dt} = 6 \\
i.e., \quad & \frac{dx}{dt} = \frac{6}{4} = \frac{3}{2}.
\end{aligned}
$$

[10] Volume of the cone is $V = \frac{1}{3}\pi r^2 h$. But we know that $h = 3r$. Therefore $V = \frac{1}{3}\pi r^2 \times 3r = \pi r^3$. Now differentiating both sides with respect to t we have

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}. \tag{3}$$

We know that the radius of the cone is increasing at a rate of 2 inches per minute. Mathematically it means that $\frac{dr}{dt} = 2$in/min.

(a) To find the rate of change of the volume when $r = 6$ inches, we put $r = 6$ and $\frac{dr}{dt} = 2$ in the equation (3). Then we obtain

$$\frac{dV}{dt} = 3\pi \times 36 \times 2 = 216\pi$$

Therefore the rate of change of volume at $r = 6$ inches is 216π cubic in/min.

(b) Similarly, when $r = 24$ inches we have

$$\frac{dV}{dt} = 3\pi \times (24)^2 \times 2 = 3456\pi.$$

Therefore the rate of change of volume at $r = 24$ inches is 3456π cubic in/min.

[14] Suppose the length of each edge of the cube is x cm. [Since the cube is expanding, the edge length x is also increasing. Therefore x depends on 'time' (t) i.e., x is a function of time (t).] Notice that there are total six surface planes of a cube, and area of each surface plane is x^2 (because edge length is x). Therefore total surface area of the cube is given by $S = 6x^2$.

We want to find the rate of change of surface area with respect to time i.e., we have to find $\frac{dS}{dt}$. Differentiating both side of the equation $S = 6x^2$ with respect to t (t stands for time), we have

$$\frac{dS}{dt} = 12x\frac{dx}{dt}. \tag{4}$$

We know that edges are expanding at the rate of 3cm/sec. In other words, $\frac{dx}{dt} = 3\text{cm/sec}$.

(a) Now putting $x = 1$ and $\frac{dx}{dt} = 3$ in equation (4) we get

$$\frac{dS}{dt} = 12 \times 3 = 36.$$

Therefore the surface area of the cube is increasing at the rate of 36 square cm/sec when each edge is 1 cm.

(b) Similarly, putting $x = 10$ and $\frac{dx}{dt} = 3$ in equation (4) we get

$$\frac{dS}{dt} = 120 \times 3 = 360.$$

Therefore the surface area of the cube is increasing at the rate of 360 square cm/sec when each edge is 10 cm.

[16] Given equation $y = \frac{1}{1+x^2}$. Rewrite this equation as $y = (1 + x^2)^{-1}$. Differentiating both sides with respect to t we have

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left[(1+x^2)^{-1}\right] \\ &= -(1+x^2)^{-2}\frac{d}{dt}[1+x^2] \\ &= -(1+x^2)^{-2}2x\frac{dx}{dt}. \end{aligned} \tag{5}$$

We know that $\frac{dx}{dt} = 2$ cm/min.

(a) Plugging in $x = -2$ and $\frac{dx}{dt} = 2$ in (5) we have

$$\begin{aligned} \frac{dy}{dt} &= -(1+4)^{-2} \times (-4) \times 2 \\ &= -\frac{1}{5^2} \times (-8) \\ &= \frac{8}{25} \text{ cm/min.} \end{aligned}$$

(b) Similarly, plugging in $x = 2$ and $\frac{dx}{dt} = 2$ in (5) we have

$$\frac{dy}{dt} = -\frac{8}{25} \text{ cm/min.}$$

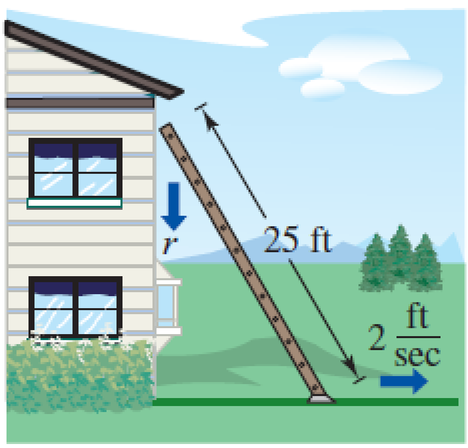
(c) When $x = 0$

$$\frac{dy}{dt} = -(1+0)^{-2} \times 0 = 0 \text{ cm/min}.$$

(d)and when $x = 10$

$$\begin{aligned} \frac{dy}{dt} &= -(1+100)^{-2} \times 20 \times 2 \\ &= -\frac{40}{101^2} \text{ cm/min}. \end{aligned}$$

[17] Let r be the height of the ladder along the wall (see figure), and the base of the ladder be at a distance x from the wall. By Pythagorean theorem we have $r^2 + x^2 = 25^2$.



We know that base of the ladder is pulled away from the house at a rate of 2 feet per second i.e., $\frac{dx}{dt} = 2$ ft/sec. We want to find how fast the top of the ladder is moving down i.e, we have to find $\frac{dr}{dt}$. Now differentiating both sides of $r^2 + x^2 = 25^2$ with respect to t we get

$$\begin{aligned} 2r\frac{dr}{dt} + 2x\frac{dx}{dt} &= 0 \\ \text{i.e.,} \qquad r\frac{dr}{dt} + x\frac{dx}{dt} &= 0 \end{aligned} \tag{6}$$

(a) When $x = 7$, we have $r = \sqrt{25^2 - 7^2} = 24$. Putting $x = 7$, $r = 24$, and $\frac{dx}{dt} = 2$ in the equation (6) we obtain

$$\begin{aligned} 24\frac{dr}{dt} + 14 &= 0 \\ \text{i.e.,} \qquad \frac{dr}{dt} &= -\frac{14}{24} = -\frac{7}{12}. \end{aligned}$$

Therefore when the base if 7 feet away from the house, the top of the ladder is moving down at the rate of $\frac{7}{12}$ ft/sec.

(b) Similarly, when $x = 15$ we have $r = \sqrt{25^2 - 15^2} = 20$. Putting $x = 15$, $r = 20$, and $\frac{dx}{dt} = 2$ in the equation (6) we obtain

$$20\frac{dr}{dt} + 30 = 0$$

$$i.e., \qquad \frac{dr}{dt} = -\frac{30}{20} = -\frac{3}{2}.$$

Therefore when the base if 15 feet away from the house, the top of the ladder is moving down at the rate of $\frac{3}{2}$ ft/sec.

(c) When $x = 24$, we have $r = \sqrt{25^2 - 24^2} = 7$. Putting $x = 24$, $r = 7$, and $\frac{dx}{dt} = 2$ in the equation (6) we obtain

$$7\frac{dr}{dt} + 48 = 0$$

$$i.e., \qquad \frac{dr}{dt} = -\frac{48}{7}.$$

Therefore when the base if 24 feet away from the house, the top of the ladder is moving down at the rate of $\frac{48}{7}$ ft/sec.

[19] Solved in class.