Section 1.4

[3] Yes it does. Because y can be uniquely written in terms of x

$$y = \frac{1}{12}x + \frac{1}{2}.$$

[7] No, because there are two possible solutions for y, namely

$$y = \pm\sqrt{x^2 - 1}.$$

[31] No, fails in vertical line test. Namely we can put a vertical line at origin, and it cuts the graph at two points $(0, 3)$ and $(0, -3)$.

[32] Yes.

[33] Yes.

[37] Provided $f(x) = x^2 + 1$ and $g(x) = x - 1$.

$$\begin{aligned}
&f(x) + g(x) = x^2 + x \\
&f(x) \cdot g(x) = (x^2 + 1)(x - 1) \\
&\frac{f(x)}{g(x)} = \frac{x^2 + 1}{x - 1} \\
&f[g(x)] = g(x)^1 + 1 = (x - 1)^2 + 1 = x^2 - 2x + 2 \\
&g[f(x)] = f(x) - 1 = x^2.
\end{aligned}$$

[40] Provided $f(x) = \frac{x}{x+1}$ and $g(x) = x^3$.

$$\begin{aligned}
&f(x) + g(x) = \frac{x}{x+1} + x^3 = \frac{x^4 + x^3 + x}{x + 1} = \frac{x(x^3 + x^2 + 1)}{x + 1} \\
&f(x) \cdot g(x) = \frac{x^4}{x + 1} \\
&\frac{f(x)}{g(x)} = \frac{1}{x^2(x + 1)} \\
&f[g(x)] = \frac{g(x)}{g(x) + 1} = \frac{x^3}{x^3 + 1} \\
&g[f(x)] = [f(x)]^3 = \left(\frac{x}{x + 1}\right)^3.
\end{aligned}$$

Domain of $f[g(x)]$ is $x \neq -1$ i.e., all real numbers except -1.

[51] Let $g(x)$ be the inverse of $f(x)$. We have $f[g(x)] = x$, which gives us

$$\begin{aligned} & 2g(x) - 3 = x \\ \Rightarrow \quad & 2g(x) = x + 3 \\ \Rightarrow \quad & g(x) = \frac{x+3}{2}. \end{aligned}$$

Also check that

$$\begin{aligned} g[f(x)] &= \frac{f(x)+3}{2} \\ &= \frac{(2x-3)+3}{2} \\ &= \frac{2x}{2} = x. \end{aligned}$$

Therefore $g(x) = \frac{x+3}{2}$ is the inverse of the function $f(x)$.

[55] Let $g(x)$ be the inverse of $f(x)$. We have $f[g(x)] = x$, which gives us

$$\begin{aligned} & \sqrt{9 - g(x)^2} = x \\ \Rightarrow \quad & 9 - g(x)^2 = x^2 \\ \Rightarrow \quad & g(x)^2 = 9 - x^2 \\ \Rightarrow \quad & g(x) = \pm\sqrt{9 - x^2}. \end{aligned}$$

Now we need to determine whether $g(x) = \sqrt{9-x^2}$ or $g(x) = -\sqrt{9-x^2}$ (note that it can't be both at a time, then it will not be a function). We know that the domain of $f(x)$ is $0 \le x \le 3$. We know that $f[g(x)] = x$. Therefore $g(x)$ can be considered as a input of $f(x)$ i.e., value of $g(x)$ must lie in the interval $[0, 3]$. Which forces $g(x)$ to be non-negative. Therefore

$$g(x) = \sqrt{9 - x^2}.$$

Now we need to determine the domain of $g(x)$. First of all, $\sqrt{9-x^2}$ is defined for $9 - x^2 \ge 0$ i.e., $-3 \le x \le 3$. But we also have

$$g[f(x)] = x \quad \text{(check that)}$$

Therefore possible inputs of $g(x)$ are $f(x)$. And we notice that $f(x) = \sqrt{9-x^2}$ is always non-negative. In fact possible outcomes of $f(x)$ i.e., range of $f(x)$ is the interval $[0, 3]$ (check that). So domain of $g(x)$ is $0 \le x \le 3$. Finally we have

$$f^{-1}(x) = g(x) = \sqrt{9 - x^2}; \quad 0 \le x \le 3.$$

Remark: If f is an invertible function then we have

$$\begin{aligned} \text{Domain} f &= \text{Range} f^{-1} \\ \text{Domain} f^{-1} &= \text{Range} f. \end{aligned}$$