MAT 16A Max time: 1 hr 30 mins **Final Exam**

Type your name here:____

Signature:____ August 2, 2013

Before the exam begins:

- Fill in all boxes above.
- Turn off all electronics and keep them out of sight: no cellular phones, iPods, wearing of headphones, not even to tell time (and not even if it's just in airplane mode).

As soon as the exam starts:

- Take a quick breath to relax! If you have truly worked through all the homework problems then you will do fine!
- check that you have all 8 pages of the exam. (The number of pages includes this cover page.)
- Write your name at the top (in the space provided) of **EACH** page. (This is to ensure your pages can be identified with you if the staple holding your exam stops working.)

During the exam:

- Keep your eyes on your own exam!
- No notes/books or electronics AT ALL!

Note that the exam length is exactly 1 hr 30 mins. When you are told to stop, you must stop **IMMEDIATELY**. This is in fairness to all students. Do not think that you are the exception to this rule.

Problem	1	2	3	4	5	6	**Total**
Score							

Print your initials:

Trigonometry

- Formula:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b, \quad \sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\sin^2 x + \cos^2 x = 1$$
$$\sec^2 x - \tan^2 x = 1$$
$$\csc^2 x - \cot^2 x = 1.$$

- Particular values:

	$x = 0$	$x = \frac{\pi}{6}$	$x = \frac{\pi}{4}$	$x = \frac{\pi}{3}$	$x = \frac{\pi}{2}$	$x = \pi$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	DNE	0
$\csc x$	DNE	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	DNE
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	DNE	-1
$\cot x$	DNE	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	DNE

Geometry

- *Circle:* If radius=r, then perimeter=$2\pi r$ and area=πr^2.
- *Sphere:* If radius=r, then surface area=$4\pi r^2$ and volume=$\frac{4}{3}\pi r^3$.
- *Solid:* Volume of a rectangular solid is length×width×height.
- *Cone:* If height=h and radius (of circular base)=r, then volume=$\frac{1}{3}\pi r^2 h$.

Logarithm

$$\log ab = \log a + \log b, \quad \text{for } a > 0, b > 0$$
$$\log \frac{a}{b} = \log a - \log b, \quad \text{for } a > 0, b > 0$$
$$\log a^b = b \log a, \quad \text{for } a > 0$$

$$e^{\ln a} = a, \quad \text{for } a > 0$$
$$\ln e^a = a, \quad \text{true for any real number } a$$
$$\ln e = 1$$
$$\ln 1 = 0.$$

Print your initials:

Problem 1: (10 points) Find the values of a and b such that the following function becomes continuous on the whole real line.

$$f(x) = \begin{cases} 3x + a, & \text{if } x < 1 \\ -x^3 + 4, & \text{if } 1 \leq x \leq 2 \\ bx^2 + 2 & \text{if } x > 2 \end{cases}$$

Solution: Possible points of discontinuities are $x = 1, 2$. We want the function to be continuous at both $x = 1$ and $x = 2$. We compute

$$\begin{aligned} &\lim_{x \to 1^-} f(x) = 3 + a \\ &\lim_{x \to 1^+} f(x) = -1^3 + 4 = -1 + 4 = 3 \\ &\lim_{x \to 2^-} f(x) = -2^3 + 4 = -8 + 4 = -4 \\ &\lim_{x \to 1^+} f(x) = 4b + 2. \end{aligned}$$

Since the function has to be continuous at $x = 1$ and $x = 2$ we must have

$$\begin{aligned} & \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \\ i.e., \quad & 3 + a = 3 \\ i.e., \quad & a = 0, \end{aligned}$$

and

$$\begin{aligned} & \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \\ i.e., \quad & -4 = 4b + 2 \\ i.e., \quad & 4b = -6 \\ i.e., \quad & b = -\frac{6}{4} = -\frac{3}{2}. \end{aligned}$$

Print your initials:

Problem 2: (10 points) Find the equation of the tangent line to the graph of $f(x) = 3 + x\cos x$ at $(0, 3)$.

Solution: Slope of the tangent line is $f'(0)$. Differentiating f with respect to x we get

$$\begin{aligned} f'(x) &= 0 + \frac{d}{dx}[x]\cos x + x\frac{d}{dx}[\cos x] \\ &= \cos x - x\sin x. \end{aligned}$$

Therefore slope of the tangent line is $f'(0) = \cos 0 - 0 = 1$. Hence equation of the tangent line is

$$\begin{aligned} & y - 3 = f'(0)(x - 0) \\ i.e., \quad & y - 3 = x \\ i.e., \quad & y - x - 3 = 0. \end{aligned}$$

Print your initials:

Problem 3: (10 points) Find the derivative of $f(x) = \frac{\tan\sqrt{x}}{1+x^2}$, $x \geq 0$.

Solution: Using quotient rule and chain rule we have

$$\begin{aligned} f'(x) &= \frac{(1+x^2)\frac{d}{dx}[\tan\sqrt{x}] - \tan\sqrt{x}\frac{d}{dx}[1+x^2]}{(1+x^2)^2} \\ &= \frac{(1+x^2)\sec^2\sqrt{x}\frac{d}{dx}[\sqrt{x}] - 2x\tan\sqrt{x}}{(1+x^2)^2} \\ &= \frac{\frac{1}{2\sqrt{x}}(1+x^2)\sec^2\sqrt{x} - 2x\tan\sqrt{x}}{(1+x^2)^2} \\ &= \frac{(1+x^2)\sec^2\sqrt{x} - 4x\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}(1+x^2)^2}. \end{aligned}$$

Print your initials:

Problem 4: (10 points) A point is moving along the graph of $y = x^3$ such that $\frac{dy}{dt} = 2$ centimeters per minute. Find $\frac{dx}{dt}$ when $x = 1$.

Solution: Differentiating the equation $y = x^3$ with respect to t we get

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt}.$$

We know that $\frac{dy}{dt} = 2$ cm/sec. Therefore $\frac{dx}{dt}$ at $x = 1$ is

$$2 = 3\frac{dx}{dt}$$

$$\textit{i.e.,} \qquad \frac{dx}{dt} = \frac{2}{3} \text{ cm/sec.}$$

Print your initials:

Problem 5: (10 points) Find all relative extrema of $f(x) = 2x^3 - 3x^2 - 12x$.

Solution: Differentiating f twice with respect to x we have

$$f'(x) = 6x^2 - 6x - 12,$$

and

$$f''(x) = 12x - 6.$$

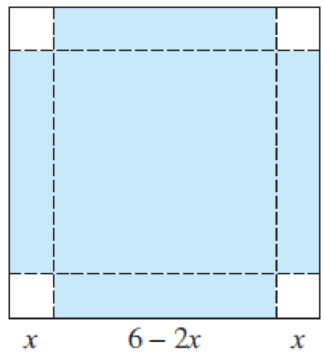
To find the critical points we solve

$$\begin{aligned} & f'(x) = 0 \\ i.e., \quad & 6x^2 - 6x - 12 = 0 \\ i.e., \quad & x^2 - x - 2 = 0 \\ i.e., \quad & (x+1)(x-2) = 0 \\ i.e., \quad & x = -1, 2. \end{aligned}$$

We notice that $f''(-1) = -18 < 0$ and $f''(2) = 18 > 0$. Therefore $x = -1$ is a maxima and $x = 2$ is a minima.

Print your initials:

Problem 6: (10 points) An open box is to be made from a six-inch by six-inch square piece of material by cutting equal squares from the corners and turning up the sides. Find the volume of the largest box that can be made.



Solution: Let side length of the cut out squares is x in. Then the base area of the open top box is $(6-2x)\times(6-2x) = (6-2x)^2$, and height of the box is going to be x. Hence volume of the open top box is going to be

$$\begin{aligned} V &= (6-2x)^2 x \\ &= (4x^2 - 24x + 36)x \\ &= 4x^3 - 24x^2 + 36x. \end{aligned}$$

We want to maximize the volume. Differentiating V twice with respect to x we get

$$\frac{dV}{dx} = 12x^2 - 48x + 36,$$

and

$$\frac{d^2V}{dx^2} = 24x - 48.$$

To find the critical numbers we solve

$$\begin{aligned} & \frac{dV}{dx} = 0 \\ i.e., \quad & 12x^2 - 48x + 36 = 0 \\ i.e., \quad & x^2 - 4x + 3 = 0 \\ i.e., \quad & (x-1)(x-3) = 0 \\ i.e., \quad & x = 1, 3. \end{aligned}$$

We test

$$\left.\frac{d^2V}{dx^2}\right|_{x=1} = -24 < 0 \quad \text{and} \quad \left.\frac{d^2V}{dx^2}\right|_{x=3} = 24 > 0.$$

Therefore $x = 1$ is a maxima. Hence the largest volume that can be made is

$$\begin{aligned} V_{max} &= (6-2)^2 \cdot 1 \\ &= 16 \text{ square inch.} \end{aligned}$$